

On the removal lemma for linear configurations in finite abelian groups

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Abstract

We present a general framework to represent discrete configuration systems using hypergraphs. This representation allows one to transfer combinatorial removal lemmas to their analogues for configuration systems. These removal lemmas claim that a system without many configurations can be made configuration-free by removing a few of its constituent elements. As applications of this approach we give; an alternative proof of the removal lemma for permutations by Cooper [6], a general version of a removal lemma for linear systems in finite abelian groups, an interpretation of the mentioned removal lemma in terms of subgroups, and an alternative proof of the counting version of the multidimensional Szemerédi theorem in abelian groups with generalizations.

1 Introduction

In 1976 Ruzsa and Szemerédi introduced the Triangle Removal Lemma [28], which roughly states that in a graph with not many triangles (copies of the complete graph on 3 vertices), we can remove a small proportion of the edges to leave the graph with no copies of the triangle at all.¹ They gave a short proof of Roth’s theorem [27] as an application: any subset of the integers with positive (upper) density contains non-trivial 3-term arithmetic progressions.² The case for k -term arithmetic progressions, conjectured by Erdős and Turán [38], was established by Szemerédi in 1975 and is now called Szemerédi’s Theorem.

Both the Triangle Removal Lemma and Szemerédi’s Theorem use Szemerédi’s Regularity Lemma [35] in their original proofs.³ However, Szemerédi’s Theorem does not seem to easily follow from the Triangle Removal Lemma (or the general Removal Lemma for Graphs [8, 11].) Indeed, while the Triangle Removal Lemma follows almost immediately from the Regularity Lemma, the proof of Szemerédi’s Theorem is more involved and the Regularity Lemma is used in only one, yet crucial, step.

In [10] Frankl and Rödl showed that a k -uniform hypergraph version of the Removal Lemma suffices to establish the existence of non-trivial $(k + 1)$ -term arithmetic progressions in subsets of the integers with positive density.⁴ The argument is similar to the one used by Ruzsa and Szemerédi [28] to show the 3-term case using the Triangle Removal Lemma.⁵ This simple proof of Szemerédi’s Theorem, along with the many applications that the Regularity Lemma for Graphs has (see the surveys [18, 17]), were the main motivations to extend the Regularity and the Removal lemmas from graphs to hypergraphs. This extension was done by several authors following different approaches:

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¹Recall that a (hyper)graph is composed by an ordered pair of sets; the vertex set and the set of edges, which are subsets of vertices. A hypergraph is said to be k -uniform if all the edges are subsets of k vertices. A graph is a 2-uniform hypergraph. A copy of a triangle is an injective map from $\{1, 2, 3\}$ to the vertex set of the graph that sends an edge of the triangle to an edge in the graph. To be precise, Ruzsa and Szemerédi showed the $(6, 3)$ -Theorem, which states the following: in any 3-uniform hypergraph with δn^2 edges, there are 6 vertices spanning (or inducing) 3 edges if $n > n_0(\delta)$.

²The k -term arithmetic progressions are the configurations $\{a, a + d, \dots, a + (k - 1)d\}$ with $a \in \mathbb{Z}$ and $d \in \mathbb{Z}_{\geq 0}$.

³Szemerédi’s Regularity Lemma states that the vertex set of any graph can be partitioned into finitely many parts, such that between most of the pairs, we have a quasi-random bipartite graph.

⁴The Removal Lemma for Hypergraphs states that if a given k -uniform hypergraph K has few copies of a fixed k -uniform hypergraph H , then it can be made free of copies of H by removing few edges. See Theorem 11.

⁵In this paper, the argument of Ruzsa and Szemerédi can be found in the proof of Theorem 6 but the construction of the hypergraph and the use of the Removal Lemma for Hypergraphs has been substituted by the removal lemma for homomorphisms Theorem 2.

a combinatorial approach from Nagle, Rödl, Schacht, and Skokan [23, 25], an approach using quasirandomness by Gowers [14], a probabilistic approach from Tao [36], and a non-standard measure-theoretic approach by Elek and Szegedy [7].

1.1 Arithmetic removal lemmas

In [15], Green used Fourier analysis techniques to establish a regularity lemma and a removal lemma for linear equations in finite abelian groups. These statements are analogous to their combinatorial counterparts. The Group Removal Lemma [15, Theorem 1.5] ensures that for every $\varepsilon > 0$ and every positive integer m , there exists a $\delta = \delta(\varepsilon, m) > 0$ such that, for any finite abelian group G , if

$$x_1 + \cdots + x_m = 0 \tag{1}$$

has less than $\delta|G|^{m-1}$ solutions with $x_i \in X_i \subset G$, then we can, by removing less than $\varepsilon|G|$ elements in each X_i , create sets X_i'' for which there is no solution to (1) with $x_i \in X_i''$ for all $i \in [1, m]$.

Král', Serra and the author [20] gave an alternative proof of the above removal lemma, showed by Green as [15, Theorem 1.5], using the removal lemma for directed graphs [1]. With this alternative approach, the result was extended to any finite group, eliminating the need of commutativity.

While a removal lemma for linear systems for some 0/1-matrices was shown to hold in [20] using graphs, the work of Frankl and Rödl [10] suggested that the hypergraph setting might provide the right tools to extend the removal lemma for one equation to a linear system. Indeed, Shapira [32], and independently Král', Serra, and the author [22], used the Removal Lemma for Hypergraphs to obtain a removal lemma for linear systems over finite fields and proved a conjecture by Green [15, Conjecture 9.4] regarding a removal lemma for linear systems in the integers. A partial result for finite fields was obtained by Král', Serra, and the author [19], and also independently by Candela [3].

In addition to showing the removal lemma for finite fields, Shapira [32] raised the issue of whether an analogous result holds for linear systems over finite abelian groups. In [21], Král', Serra, and the author answered the question affirmatively provided that the determinantal⁶ of the integer matrix that defines the system is coprime to the cardinality of the finite abelian group. See [21, Theorem 1] or Theorem 3 for the result.

In a different direction, Candela and Sisask [4] proved that a removal lemma for integer linear systems holds over certain compact abelian groups. The main result in [4] has been recently extended by Candela, Szegedy and the author [5] to any compact abelian group provided that the integer matrix has determinantal 1.

Previous combinatorial arguments. The proof schemes of the previous arithmetic removal lemmas in [20, 22, 21, 32] are inspired by the approaches of [28] and [10], and can also be found in [14, 33, 34, 37]. Concisely, the main argument involves constructing a pair of graphs (or hypergraphs) (K, H) so as to make it possible to transfer the removal lemma from the graph/hypergraph combinatorial setting to an arithmetic context. The pair (K, H) is said to be a representation of the system and usually satisfies the property that each copy of H in K is associated with a solution of our system. Moreover, these copies of H should be evenly distributed throughout the edges of K . The notion of representability of a system by a hypergraph has been formalized by Shapira in [32, Definition 2.4].

1.2 Main notions and results

This paper is built around two main pieces. The first one is the generalization of the combinatorial representability notion introduced by Shapira in [32]. This generalization is stated as Definition 10. The second piece is a removal lemma for homomorphisms of finite abelian groups; given a finite abelian group G and a homomorphism A from G^m to G^k , if a set $X = X_1 \times \cdots \times X_m$ does not contain many $\mathbf{x} \in X$ with $A\mathbf{x} = 0$, then X can be made solution-free by removing a small proportion of each X_i . The detailed statement can be found below as Theorem 2. Additionally, we provide an interpretation of Theorem 2 in terms of subgroups, which is stated as Theorem 4, and can be seen as a removal lemma for finite abelian subgroups.

⁶The determinantal of order i of an integer matrix A is the greatest common divisor of all the $i \times i$ submatrices of A . In this paper, the term *determinantal* is used to refer to the determinantal of maximal order. See [24, Chapter II, Section 13] or Section 5.2 for additional details and more of its properties.

1.2.1 Systems of configurations and representability

Systems of configurations. Let us introduce the notion of a (finite) system of configurations. Let m be a positive integer and let G be a set. A system of degree m over G consists of a pair (A, G) , where A is a property on the configurations of G^m , $A : G^m \rightarrow \{0, 1\}$. If $x \in G^m$ is such that $A(x) = 1$, then x is said to be a *solution* to (A, G) . In this paper all the sets G considered are finite.

Representable systems. We introduce as Definition 10 a more general notion of representability for systems than the one given by Shapira in [32, Definition 2.4]. Although rather technical, Definition 10 asks for the existence of a pair of hypergraphs (K, H) , associated to a finite system of configurations (A, G) , with the following summary of properties (in parenthesis appear references to the properties described in Definition 10):

- Each copy of H in K is associated to a solution of (A, G) (domain and range of r in RP2.) The edges are associated with elements of G (RP1, third point.)
- Given a solution of (A, G) , there are many copies of H in K associated to such solution (cardinality of $r^{-1}(\mathbf{x}, q)$ in RP2.) Those copies are well (evenly) distributed through the edges associated to the elements that configure the solution (RP3 and RP4.)
- The number of vertices in H is bounded (first point in RP1.)

All these points are sufficient to prove a removal lemma for representable systems, which has Theorem 1 as the precise statement.

Let us mention that the representations used in [20, 22, 21, 32] can be seen to give a representation according to Definition 10. In [21], the authors used Shapira's definition, with an extra post-processing, to show a removal lemma for integer linear systems with determinantal 1 and where the sets to be removed are small with respect to the total group, [21, Theorem 1]. The additional features of Definition 10 with respect to the representation notion given by Shapira in [32] allow us to extend the result [21, Theorem 1] (Theorem 3 in this paper) to Theorem 2 in the following way.

- The set Q is used to remove the determinantal condition and to extend the result to any homomorphism A with domain G^m and image in G^k .
- The vector γ or, more precisely, the vector of proportions $(1/\gamma_1, \dots, 1/\gamma_m)$, allows us to claim that the i -th removed set is an ϵ -proportion of the projection of the whole solution set onto the i -th coordinate. The projection of the solution set is, in general, smaller than the whole abelian group G .⁷ See Theorem 1 for further details.

1.2.2 Removal lemma for representable systems.

Let $I \subset \mathbb{N}$ be a set of indices. Consider $(\mathcal{A}, \mathcal{G}, m) = \{(A_i, G_i)\}_{i \in I}$ a family of systems of degree m . Let $S(A, G)$ denote the set of solutions for $(A, G) \in (\mathcal{A}, \mathcal{G}, m)$ and let $S(A, G, X)$ denote the subset of $\mathbf{x} \in S(A, G)$ with $\mathbf{x} \in X \subset G^m$.

Let $\Gamma = \{\gamma(i)\}_{i \in I} = \{(\gamma_1(i), \dots, \gamma_m(i))\}_{i \in I}$ be a family of m -tuples of positive real numbers indexed by I . The family $(\mathcal{A}, \mathcal{G}, m)$ of systems is said to be Γ -representable, and the system (A_i, G_i) is said to be associated with γ_i , if Definition 10 in Section 2 holds. The representability property suffices to show a removal lemma for configuration systems.

Theorem 1 (Removal lemma for representable systems). *Let $(\mathcal{A}, \mathcal{G}, m)$ be a Γ -representable family of systems. Let (A, G) be an element in the family associated to $\gamma = (\gamma_1, \dots, \gamma_m)$. Let X_1, \dots, X_m be subsets of G and let $X = X_1 \times \dots \times X_m$.*

For every $\varepsilon > 0$ there exists a $\delta = \delta(m, \varepsilon) > 0$, universal for all the members of the family, such that if

$$|S(A, G, X)| < \delta |S(A, G)|,$$

⁷For instance, the projection onto the first coordinate of the solution set of the equation $x_1 + 2(x_2 + x_3) = 0$, with $x_i \in \mathbb{Z}_6$, is isomorphic to \mathbb{Z}_3 .

then there are sets $X'_i \subset X_i$ with $|X'_i| < \varepsilon|G|/\gamma_i$ for which

$$S(A, G, X \setminus X') = \emptyset,$$

where $X \setminus X' = (X_1 \setminus X'_1) \times \cdots \times (X_m \setminus X'_m)$.

Let us notice that, if a family of systems is Γ -representable, then the conclusions of Theorem 1 holds with smaller $\gamma_j(i)$'s as the restrictions on the theorem decrease with the γ 's. In the representability notion of Shapira [32], as well as in other works like [20, 22, 21], the $\gamma_j(i)$ are all 1. Thus, the notion of representability Definition 10 is an extension of [32, Definition 2.4].

1.2.3 Removal lemma for finite abelian groups.

Let G be a finite abelian group. Given $\mathbf{b} \in G^k$, a homomorphism A from G^m to G^k induces a property (A, \mathbf{b}) in G^m given by: $\mathbf{x} \in S((A, \mathbf{b}), G)$ if and only if $A(\mathbf{x}) = \mathbf{b}$. Let $(\mathcal{A}, \mathcal{G}, m)$ be the family of systems given by the homomorphisms (A, \mathbf{b}) with fixed m . These are the configuration systems that we consider for most of the paper, especially in Section 3 and onwards.

The set of homomorphisms $A : G^m \rightarrow G^k$ are in bijection with $k \times m$ homomorphism matrices $(\vartheta_{i,j})$ for some homomorphisms $\vartheta_{i,j} : G \rightarrow G$ depending on A .⁸ In particular, given $\mathbf{b} = (b_1, \dots, b_k)^\top \in G^k$, $(x_1, \dots, x_m) \in S((A, \mathbf{b}), G)$ if and only if

$$\begin{pmatrix} \vartheta_{1,1} & \cdots & \vartheta_{1,m} \\ \vdots & \ddots & \vdots \\ \vartheta_{k,1} & \cdots & \vartheta_{k,m} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \iff \sum_{i=1}^m \vartheta_{j,i}(x_i) = b_j, \forall j \in [1, k].$$

Thus, we may use the term $k \times m$ homomorphism system on G to refer to the system induced by a homomorphism from G^m to G^k .

Let $S_i((A, \mathbf{b}), G)$, $i \in [1, m]$ denote the projection of the solution set $S((A, \mathbf{b}), G)$ to the i -th coordinate of G^m , $S_i((A, \mathbf{b}), G) = \pi_i(S((A, \mathbf{b}), G))$. The solution set $S((A, \mathbf{b}), G)$ is denoted by $S(A, G)$ when $\mathbf{b} = 0$ or understood by the context.

In the sections 4 and 5 of this paper, we show that the family of homomorphisms of finite abelian groups is Γ -representable with $\gamma_i = |G|/|S_i((A, \mathbf{b}), G)|$ when $m \geq k + 2$. Hence Theorem 1, together with the additional comments to the construction presented in Section 6, implies Theorem 2.

Theorem 2 (Removal lemma for linear systems over abelian groups). *Let G be a finite abelian group and let m, k be two positive integers. Let A be a group homomorphism from G^m to G^k . Let $\mathbf{b} \in G^k$. Let $X_i \subset G$ for $i = [1, m]$, and $X = X_1 \times \cdots \times X_m$.*

For every $\varepsilon > 0$ there exists a $\delta = \delta(m, \varepsilon) > 0$ such that, if

$$|S((A, \mathbf{b}), G, X)| < \delta |S((A, \mathbf{b}), G)|,$$

then there are sets $X'_i \subset X_i \cap S_i((A, \mathbf{b}), G)$ with $|X'_i| < \varepsilon |S_i((A, \mathbf{b}), G)|$ and

$$S((A, \mathbf{b}), G, X \setminus X') = \emptyset, \text{ where } X \setminus X' = (X_1 \setminus X'_1) \times \cdots \times (X_m \setminus X'_m).$$

Also, let $I \subset [1, m]$ be such that $X_i \supset S_i((A, \mathbf{b}), G)$ for $i \in I$. The previous statement holds with the extra condition that $X'_i = \emptyset$ for $i \in I$.

Let us state the known arithmetic removal lemma for finite abelian groups [21, Theorem 1].

Theorem 3 (Removal lemma for finite abelian groups, Theorem 1 in [21]). *Let G be a finite abelian group and let m, k be two positive integers. Let A be a $k \times m$ integer matrix with determinantal coprime with the order of the group $|G|$.⁹ Let $\mathbf{b} \in G^k$. Let $X_i \subset G$ for $i = [1, m]$, and $X = X_1 \times \cdots \times X_m$. For every $\varepsilon > 0$ there exists a $\delta = \delta(m, \varepsilon) > 0$ such that, if*

$$|S((A, \mathbf{b}), G, X)| < \delta |S((A, \mathbf{b}), G)| = \delta |G^{m-k}|,$$

⁸See [39, Section 13.10]

⁹The determinantal can be assumed to be 1. This restriction implies that $|S((A, \mathbf{b}), G)| = |G^{m-k}|$, which is not true in the general case, as Observation 25, or footnote 7 shows.

then there are sets $X'_i \subset X_i \cap S_i((A, \mathbf{b}), G)$ with $|X'_i| < \varepsilon|G|$ and

$$S((A, \mathbf{b}), G, X \setminus X') = \emptyset, \text{ where } X \setminus X' = (X_1 \setminus X'_1) \times \cdots \times (X_m \setminus X'_m).$$

We can see that Theorem 2 extends Theorem 3 in three ways:

- The coprimality condition between the determinantal of A and the order of the group is not needed.
- The systems induced by homomorphisms are more general than the systems induced by integer matrices. In particular, if $G = \prod_{i=1}^t \mathbb{Z}_{n_i}$, $n_{i+1} | n_i$, and we let $x = (x_1, \dots, x_t) \in G$ with $x_i \in \mathbb{Z}_{n_i}$, then the homomorphism systems allow for linear equations between the components x_i and x_j . This fact is used to prove the multidimensional version of Szemerédi's Theorem. See Section 1.3.
- The sizes of the deleted sets X'_i are an ε -proportion of $|S_i((A, \mathbf{b}), G)|$ and not of $|G|$. In particular, if $|S_i((A, \mathbf{b}), G)| = 1$, $\varepsilon < 1$, and X_i contains that element, then $X'_i = \emptyset$. This makes the result best possible in the following sense: if $S((A, \mathbf{b}), G, X) > \delta|S((A, \mathbf{b}), G)|$, then, in order to delete all the solutions, there should exist an i with $|X'_i| > |S_i((A, \mathbf{b}), G)|\delta/m$. Therefore, we remove, at most, an ε -proportion of the right order of magnitude.
- The set of variables x_i with $X_i = S_i(A, G)$ and for which no element should be removed is arbitrary. The argument leading to [21, Theorem 1], allows the existence of a set of indices I of full sets from which no element is removed. However, the argument from [21] imposes an upper bound on the size of I . The argument in Section 5.9 remove those bounds on I .¹⁰

1.2.4 Removal lemma for finite abelian subgroups

As Theorem 2 can be applied to any finite abelian group G and any homomorphism, we can rephrase the result in terms of subgroups.

Theorem 4 (Removal lemma for subgroups). *For every $\epsilon > 0$ and every positive integer m , there exists a $\delta = \delta(\epsilon, m) > 0$ such that the following holds. Let G_1, \dots, G_m be finite abelian groups. Let S be a subgroup of $G_1 \times \cdots \times G_m$ and let $s \in \prod_{i \in [m]} G_i$. Let X_i be a subset of G_i for each $i \in [1, m]$. If $|[X_1 \times \cdots \times X_m] \cap s + S| < \delta|S|$ then there exist X'_1, \dots, X'_m , with $|X'_i| < \epsilon\pi_i(S)$ for all $i \in [1, m]$, such that $\{[X_1 \setminus X'_1 \times \cdots \times X_m \setminus X'_m] \cap s + S\} = \emptyset$. If $\pi_i(s + S) \subset X_i$, for some $i \in [1, m]$ then we can assume $X'_i = \emptyset$.*

Indeed, any subgroup of a finite abelian group is the kernel of a homomorphism (namely the quotient map $\prod_{i \in [m]} G_i \rightarrow [\prod_{i \in [m]} G_i] / S$, S being our subgroup of interest.) Notice also that, instead of $\prod_{i \in [m]} G_i$, we could consider our domain to be a supergroup $G^m > \prod_{i \in [m]} G_i$, for some suitable finite abelian group G , as S is also a subgroup of G^m . Moreover, we can assume $G^m / S < G^k$ for some k (take, for instance $k = m$.) Therefore, Theorem 2 suffices to show Theorem 4. Since the kernel of a homomorphism generates a subgroup of G^m , Theorem 4 implies Theorem 2. Hence the version of the result for subgroups Theorem 2, and the version for systems of homomorphisms Theorem 4, are equivalent.

1.2.5 Removal lemma for permutations

In [6], Cooper introduced a regularity lemma and a removal lemma for permutations. Let $\mathcal{S}(i)$ denote the set of bijective maps from $[0, i-1]$ to $[0, i-1]$. Slightly modifying the notation in [6], let $\Lambda^\tau(\sigma)$, for $\tau \in \mathcal{S}(m)$ and $\sigma \in \mathcal{S}(n)$, be the set of occurrences of the pattern τ in σ . That is to say, the set of index sets $\{x_0 < \cdots < x_{m-1}\} \subset [0, n-1]$ such that $\sigma(x_i) < \sigma(x_j)$ if and only if $\tau(i) < \tau(j)$.

Proposition 5 (Proposition 6 in [6]). *Suppose that $\sigma \in \mathcal{S}(n)$, $\tau \in \mathcal{S}(m)$. For every $\epsilon > 0$ there exist a $\delta = \delta(\epsilon, m) > 0$ such that, if $|\Lambda^\tau(\sigma)| < \delta n^m$, then we may delete at most ϵn^2 index pairs to destroy all copies of τ in σ .*

In Section 2.3.2 we can find a representation where the valid configurations are given by the set $\Lambda^\tau(\sigma)$, and where we shall delete pairs of indices to destroy them. Hence Proposition 5 follows from Theorem 1.

¹⁰The argument of Section 5.9 and of Observation 34 could be applied to [21, Theorem 1] to add this extra property.

1.3 Applications

Multidimensional Szemerédi. One of the main applications of Theorem 2 is a new proof of the counting version of the multidimensional Szemerédi's Theorem for finite abelian groups.

The original proof of the multidimensional Szemerédi theorem for the integers was given by Furstenberg and Katznelson [12] and uses ergodic theory. Solymosi [33] observed that a removal lemma for hypergraphs would imply the multidimensional Szemerédi theorem (a detailed construction can be found in [14]). Solymosi's geometric argument uses hypergraphs and follows the lines of the argument by Ruzsa and Szemerédi [28] to obtain Roth's Theorem [27] from the Triangle Removal Lemma. With the development of the Regularity Method for Hypergraphs [14, 23, 25, 36] and the corresponding Removal Lemma for Hypergraphs [14, 23, 36], a combinatorial proof of the multidimensional version of Szemerédi's Theorem for the integers could be pushed forward [14, 33].

In [37], Tao uses the same construction as Solymosi [33] to show [37, Theorem B.2] and a lifting trick to obtain a generalized version of the multidimensional Szemerédi theorem for finite abelian groups, [37, Theorem B.1]. Theorem 2 can be used to prove both [37, Theorem B.1] and [37, Theorem B.2].

The argument to deduce [37, Theorem B.2] (Theorem 6 in this paper) from Theorem 2 explicitly shows that the dependencies in [37, Theorem B.2] and in [37, Theorem B.1] are independent of the dimension of the space and depend only on the number of points required in the configuration. On the other hand, the relation between δ and ϵ obtained using Theorem 2 is worse than the direct construction of [37, Theorem B.2] due to the larger uniformity of the hypergraph used.

Theorem 6 (Multidimensional Szemerédi for finite abelian groups, Theorem B.2 in [37]). *Let $\epsilon > 0$. Let G^m be a finite abelian group and let $S \subset G^m$ be such that $|S|/|G^m| \geq \epsilon$. There exists $\delta = \delta(\epsilon, m+1) > 0$ such that the number of configurations of the type $\{(x_1, \dots, x_m), (x_1 + a, x_2, \dots, x_m), \dots, (x_1, x_2, \dots, x_m + a)\} \subset S$, for some $a \in G$, is at least $\delta|G|^{m+1}$.*

Proof of Theorem 6. Consider the abelian group $P = G^m$, $X_i = S \subset P$, for $i \in [1, m+1]$. Let $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,m})$, $i \in [1, m+1]$, be the variables of the homomorphism system that can be derived from the following linear equations:

$$\begin{cases} x_{1,1} - x_{2,1} = x_{1,j} - x_{j+1,j} & \text{for } j \in [2, m] \\ x_{1,j} = x_{i,j} & \text{for all } (i, j) \in [1, m+1] \times [1, m], i \neq j+1. \end{cases} \quad (2)$$

Indeed, \mathbf{x}_1 is thought of as the centre of the configuration. The first equations state that the difference between the j -th coordinate of \mathbf{x}_{j+1} and the j -th coordinate of \mathbf{x}_1 is the same regardless of j ; this is achieved by setting all the differences to be equal to the difference between the first coordinate of \mathbf{x}_2 and the first coordinate of \mathbf{x}_1 . The second set of equations treat the other coordinates, imposing that all the other coordinates of \mathbf{x}_{j+1} , except the j -th, should be equal to those of \mathbf{x}_1 . Therefore, $(\mathbf{x}_1, \dots, \mathbf{x}_{m+1})$ is a solution to the system defined by (2) if and only if $\mathbf{x}_1 = (y_1, \dots, y_m)$, $\mathbf{x}_2 = (y_1 + a, \dots, y_m)$, \dots , $\mathbf{x}_{m+1} = (y_1, \dots, y_m + a)$ for some $y_1, \dots, y_m, a \in G$.

By adding some trivial equations, like $0 = 0$, the system induces a homomorphism $A : P^{m+1} \rightarrow P^m$, with $S(A, P) \cong G^{m+1}$. Observe that $S_i(A, P) \cong G^m$ as any point in $P = G^m$ can be the i -th element in the configuration.

Consider the $\delta = \delta_{\text{Theorem 2}}(m+1, \epsilon/(m+1))$ coming from Theorem 2 applied with $\epsilon/(m+1)$ and $m+1$. Let us proceed by contradiction and assume that the number of solutions is less than $\delta S(A, P) = \delta|G|^{m+1}$. Now we apply Theorem 2 and find sets X'_1, \dots, X'_{m+1} with $|X'_i| < |G^m|\epsilon/(m+1)$ such that the sets $X_i = S \setminus X'_i$ bear none of the desired configurations.

Observe that any point $\mathbf{x} \in S \subset G^m$ generates a solution to the linear system as $(\mathbf{x}, \dots, \mathbf{x}) \in P^{m+1}$ is a valid configuration with $a = 0_G$. Consider $S' = S \setminus (\cup X'_i)$. Since $|S| > \epsilon|G^m|$ and $|X'_i| < |G^m|\epsilon/(m+1)$, there exists an element \mathbf{s} in S' , as S' is non-empty. Therefore $\mathbf{s} \in X_i$ for every $i \in [1, m+1]$. Thus $(\mathbf{s}, \dots, \mathbf{s}) \in P^{m+1}$ is a solution that still exists after removing the sets X'_i of size at most $\epsilon/(m+1)$ from S . This contradicts Theorem 2. Therefore, we conclude that at least $\delta|G|^{m+1}$ solutions exist. \square

Other linear configurations. More generally, we can show the following corollary of Theorem 2.

Corollary 7. *Let G be a finite abelian group, let A be a $k \times m$ homomorphism for G and let $\mathbf{b} \in G^k$. Assume that $S(A, G) = S((A, \mathbf{b}), G) \subset G^m$ contains a set R satisfying the following conditions.*

- (i) *The projection of R onto the i -th coordinate of G^m is $S_i(A, G)$. This is, $\pi_i(R) = S_i(A, G)$.*

(ii) For each $i \in [1, m]$ and for each pair $g_1, g_2 \in S_i(A, G)$, $|\pi_i^{-1}(g_1) \cap R| = |\pi_i^{-1}(g_2) \cap R|$.

Then, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, m) > 0$ such that, for any $S \subset G$ with $|S^m \cap R| > \varepsilon|R|$, we have $|S(A, G, S^m)| \geq \delta|S(A, G)|$.

Proof. We proceed by contradiction. Choose $\delta = \delta_{\text{Theorem 2}}(m, \varepsilon/(m+1))$ and assume that $|S(A, G, S^m)| < \delta|S(A, G)|$. Then there are sets X'_i , with $|X'_i| < \varepsilon/(m+1)$, such that $S(A, G, \prod_{i=1}^m S \setminus X'_i) = \emptyset$. However, by (i) and (ii) we delete at most $\varepsilon \frac{m}{m+1}|R|$ hence $R \cap \prod_{i=1}^m S \setminus X'_i \neq \emptyset$, thus $S(A, G, \prod_{i=1}^m S \setminus X'_i) \neq \emptyset$ reaching a contradiction. \square

In particular, if the linear system (A, G) satisfies $S_i(A, G) = G$ for all $i \in [1, m]$ and $(x, \dots, x) \in S(A, G)$ for each $x \in G$, then Corollary 8 shows that any set $S \subset G$ with $|S| \geq \epsilon|G|$, satisfies that $|S(A, G, S^m)| > \delta|S(A, G)|$ for some $\delta > 0$ depending on ϵ and m . That is, any set with positive density will contain a positive proportion of the solutions. Corollary 7 can be particularized as Corollary 8 which presents a perhaps more directly applicable form.

Corollary 8. Let G be a finite abelian group, let G_1, \dots, G_s be subgroups of G . Let Φ_1, \dots, Φ_t be group homomorphisms

$$\begin{aligned} \Phi_i : G_1 \times \dots \times G_s &\rightarrow G \\ (x_1, \dots, x_s) &\mapsto \Phi_i(x_1, \dots, x_s). \end{aligned}$$

For every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, t) > 0$ such that, for every $S \subset G$ with $|S| \geq \epsilon|G|$,

$$\left| \left\{ x \in G, \mathbf{x} \in \prod_{i=1}^s G_i \mid (x + \Phi_1(\mathbf{x}), \dots, x + \Phi_t(\mathbf{x})) \in S^t \right\} \right| > \delta \left| \left\{ x \in G, \mathbf{x} \in \prod_{i=1}^s G_i \mid (x + \Phi_1(\mathbf{x}), \dots, x + \Phi_t(\mathbf{x})) \in G^t \right\} \right|.$$

Proof. Consider $R = \{(x, \dots, x)\}_{x \in G}$. Observe that the configuration set $\{x \in G, \mathbf{x} \in \prod_{i=1}^s G_i \mid (x + \Phi_1(\mathbf{x}), \dots, x + \Phi_t(\mathbf{x})) \in G^t\}$ is a subgroup of G^t , whence there exists a homomorphism A such that $S(A, G)$ is the configuration set and, in this case, $S_i(A, G) = G = \pi_i(R)$ for all i . Thus the hypotheses of Corollary 7 are fulfilled and the result follows. \square

Corollary 8 encompasses the simplex-like configurations from the multidimensional version of Szemerédi's theorem with the evaluation: $G = \mathbb{Z}_p^k$, $G_1 = \mathbb{Z}_p$, $\Phi_t = 0$ and Φ_1, \dots, Φ_k being the coordinate homomorphisms

$$\begin{aligned} \Phi_i : \mathbb{Z}_p &\rightarrow \mathbb{Z}_p^t \\ x &\mapsto (0, \dots, 0, \overbrace{x}^i, 0, \dots, 0). \end{aligned}$$

Additionally Corollary 8 generalizes [37, Theorem B.1] which asserts that given a finite abelian group G , for every $\epsilon > 0$ and t, m positive integers, there are, in any set $S \subset G^m$ with $|S| > \epsilon|G|^m$, $\delta(\epsilon, t, m)|G|^{m+1}$ configurations

$$\{y \in G^m, x \in G \mid (y + \Phi_1(x), \dots, y + \Phi_{(2t+1)^m}(x)) \in S^{(2t+1)^m}\}$$

with

$$\Phi_i(x) = (\chi_1(i)x, \dots, \chi_m(i)x)$$

where $(\chi_1(i), \dots, \chi_m(i))$ are the components of i in base $2t+1$ shifted by $-t$ so their values lie in $[-t, t]$ instead of the usual $[0, 2t]$. An example of an extra configuration that Corollary 8 covers are the “rectangles” $(x, x + x_1, x + x_2, x + x_1 + x_2) \in S^4$, for $S \subset G = \mathbb{Z}_3^n$, with $x_1 \in G_1$ and $x_2 \in G_2$ two subgroups of G , isomorphic to $\mathbb{Z}_3^{n-\log(n)}$ and $\mathbb{Z}_3^{\sqrt{n}}$ respectively, and such that $G_1 + G_2 = G$.

The arguments to show Corollary 8, Corollary 7, or Theorem 6 exemplify that Theorem 2 presents a comprehensive approach to the asymptotic counting of homothetic-to-a-point structures found in dense sets of products of finite

abelian groups. More precisely, the constants involved in the lower bound of the number of configurations depend only on the number of points of the configuration and on the density of the set, but not on the configuration itself nor on the structure of the finite abelian group.

If we ask for configurations in the integers, the constant does depend on the configuration as we are not interested in solutions that occur due to the cyclic nature of the components of the finite abelian group. Therefore, we should reduce the density of the sets to allow only the desired solutions. This affects the total number of configurations found in the finite abelian group.

Monochromatic solutions. Theorem 2 also allows us to extend the results in [29] regarding a counting statement for the monochromatic solutions of bounded torsion groups. In particular, we ensure that there are $\Omega(|S(A, G)|)$ monochromatic solutions, thus improving the asymptotic behaviour $\Omega(|G|^{m-k})$ stated in [29]. Here $S(A, G)$ represents the solution set of $A\mathbf{x} = 0$, $\mathbf{x} \in G^m$, when A is a $k \times m$ full rank integer matrix and the asymptotic behaviour depends on the number of colours.

Hypergraph containers. Using the hypergraph containers tools from [30], Theorem 2 can be used to extend [30, Theorem 10.3] or [31, Theorem 2.10], regarding the number of subsets free of solutions of a given system of equations, and show for instance Theorem 9, where homomorphism systems are considered. Following the notation in [30], a homomorphism system A is said to be *full rank* if there exists a solution to $Ax = \mathbf{b}$ for any $\mathbf{b} \in G^k$. A full rank $k \times m$ homomorphism system A (or with coefficients over a finite field) is said to be *abundant* if any $k \times m - 2$ subsystem of A formed using $m - 2$ columns of homomorphisms also has full rank. Given a set $Z \subset G^m$ of discounted solutions and $\mathbf{b} \in G^k$, a set $X \subset G$ is said to be Z -solution-free if there is no $x \in X^m - Z$ with $Ax = \mathbf{b}$. Let $\text{ex}(A, \mathbf{b}, G)$ denote the size of the maximum Z -solution-free set.

Theorem 9 (Saxton, Thomason, Theorem 10.3/2.10 in [30]/[31] with Theorem 2). *Let $\{G_i\}_{i \in I}$ be a sequence of finite abelian groups. Let A_i be a sequence of abundant $k \times m$ homomorphism systems and $\mathbf{b}_i \in G_i^k$ a sequence of independent vectors such that $|S((A_i, \mathbf{b}_i), G_i)| = |G_i|^{m-k}$. Let Z be such that $Z \subset S((A_i, \mathbf{b}_i), G_i)$ and $|Z| = o(|G_i|^{m-k})$. Then the number of Z -solution-free subsets of G_i is $2^{\text{ex}(A_i, \mathbf{b}_i, G_i) + o(|G_i|)}$.*

1.4 Outline of the paper

The main results of the paper are Theorem 1 and Theorem 2. To prove Theorem 1, we observe that the notion of representation, Definition 10, is sufficient to transfer the hypergraph removal lemma, Theorem 11 in this paper, to the representable setting. The argument can be found in Section 2. Some examples of representable systems and their correspondent removal lemmas are presented.

In Section 3 we introduce the notion of μ -equivalent linear systems (see Section 3.1). In Section 3.2, we show some relations between the representability of the systems (A_1, G_1) and (A_2, G_2) whenever (A_2, G_2) is μ -equivalent to (A_1, G_1) . These results are used in the proof of Theorem 2. Indeed, the strategy of the proof can be summarized as finding a suitable sequence of μ -equivalent systems, from the system of our interest, to a representable one. As Section 3.2 shows, we can then find a representation for our original system.

In Section 4 and Section 5 we prove Theorem 2 by arguing that the systems involved in the statement of the theorem are representable. Section 6 is devoted to show the cases where $m \leq k + 1$ and to prove the second part of the result involving the sets X_i for which $X_i \supset \pi_i(S(A, G))$.

The sketch of the construction for the representation is as follows. Given $G = \prod_{i=1}^t \mathbb{Z}_{n_i}$ with $n = n_1$ and $n_i | n_j$ for $i \geq j$, we interpret the homomorphism $A : G^m \rightarrow G^t$ as a homomorphism A' from $(\mathbb{Z}_n^t)^m$ to $(\mathbb{Z}_n^t)^k$ in a natural way. Then any solution of $S(A, G)$ is related to $|S(A', \mathbb{Z}_n^t)|/|S(A, G)|$ solutions of $S(A', \mathbb{Z}_n^t)$. This reduction process is detailed in Section 4.

As Section 5.1 shows, the homomorphism matrix A' can be thought of as an integer matrix from \mathbb{Z}_n^{tm} to \mathbb{Z}_n^{tk} with tm variables and tk equations in \mathbb{Z}_n . This interpretation as an integer matrix allows for the construction of the representation by using the ideas in the proof of [21, Lemma 4]. The construction is detailed in Section 5 and involves several transformations to the pair (A', G') to address the different issues like the determinantal being larger than 1. The main characteristics of those transformations are described in the statements of Section 3.2. The Γ -representability with $\gamma_j(i) > 1$ involves the generation of several systems. The construction of such systems

is detailed in Section 5.5 and they are combined in Section 5.8 to create a single 1-strongly-representable system.¹¹ A summary of all the transformations can be found in a table in Section 5.10.

2 Representable systems

In this work $[a, b]$ stands for the integers between a and b , both included. If $\mathbf{x} \in G^m$, then $(\mathbf{x})_i$ denotes the i -th component of \mathbf{x} . Let us recall some notions regarding hypergraphs. Given a hypergraph $K = (V, E)$, $V = V(K)$ denotes the vertex set, $E = E(K)$ denotes the edge set and $|K| = |V(K)|$ denotes the size of the vertex set. A hypergraph K with vertex set $V = V(K)$ and edge set $E = E(K)$ is said to be s -uniform if each edge in E contains precisely s vertices. Throughout this paper, we consider hypergraphs with edges coloured by integers. A hypergraph K is said to be m -coloured if each edge in K bears a colour in $[1, m]$. If K is an m -coloured hypergraph, $E_i(K)$ denotes the set of edges coloured $i \in [1, m]$ in K . By a copy of H in K we understand an injective homomorphism of colored hypergraphs of H into K respecting the colors of the edges (the map is from vertices to vertices, injective, and maps edges colored i to edges with color i). We use $C(H, K)$ to denote the set of colored copies of H in K . If H has m edges $\{e_1, \dots, e_m\}$ with e_i colored i then H can be identified with (e_1, \dots, e_m) .

2.1 Representability

The definition of a representable system, Definition 10, is a generalized notion of the one formalized in [32] that suffices to obtain a removal lemma; in our case Theorem 1. These representability notions have been used in several works like [3, 19, 20, 22, 21, 32, 34] to translate the conclusion of the removal lemma for graphs or hypergraphs to linear systems of equations. The representable system notion could potentially be used in more general contexts than the homomorphism systems described in this work.

Recall that a system (A, G) is a pair given by a finite set G and a property $A : G^m \rightarrow \{0, 1\}$. $S(A, G)$ denotes the preimage of 1 by A . γ denotes a tuple of m positive real numbers $(\gamma_1, \dots, \gamma_m)$. $(\mathcal{A}, \mathcal{G}, m)$ denotes a family of systems and Γ a collection of γ 's, one for each system.

Definition 10 ((strongly) representable system). *The family of finite systems $(\mathcal{A}, \mathcal{G}, m)$ is said to be Γ -representable if there are positive real numbers χ_1, χ_2 , depending on the family $(\mathcal{A}, \mathcal{G}, m)$, and for each $(A, G) \in (\mathcal{A}, \mathcal{G}, m)$ and the $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma$ associated with (A, G) , there exists a pair of coloured hypergraphs (K, H) with the following properties RP1, RP2 and RP3.*

- RP1
- K and H are s -uniform m -colored hypergraphs.
 - H has m different edges $\{e_1, \dots, e_m\}$ and the edge e_i is coloured i . Moreover $\chi_1 \geq |V(H)| = h > s \geq 2$.
 - Each edge in K bears a label in G given by $l : E(K) \rightarrow G$.

RP2 *There exist a positive integer p , a set Q , and a surjective map r*

$$\begin{aligned} r : C(H, K) &\longrightarrow S(A, G) \times Q \\ H = \{e_1, \dots, e_m\} &\longmapsto (r_0(H), r_q(H)) \end{aligned}$$

such that $r_0(H) = (l(e_1), \dots, l(e_m))$, and, for any given $\mathbf{x} \in S(A, G)$ and $q \in Q$, the set $r^{-1}(\mathbf{x}, q)$ has size

$$|r^{-1}(\mathbf{x}, q)| = p\lambda \prod_{i=1}^m \gamma_i \text{ with } \lambda = c \frac{|K|^s}{|G|}$$

for some $c \geq \chi_2$.

RP3 *If e_i is an edge coloured i in a copy $H \in r^{-1}(\mathbf{x}, q)$, then $p \frac{\prod_{j=1}^m \gamma_j}{\gamma_i}$ copies of H in $r^{-1}(\mathbf{x}, q)$ contain e_i .*

If, additionally,

RP4 *For any edge e_i coloured i and $l(e_i) = \mathbf{x}_i$, there exists a copy of $H \in r^{-1}(\mathbf{x}, q)$, with $(\mathbf{x})_i = \mathbf{x}_i$, containing e_i ,*

¹¹See Definition 10 for the additional conditions of the strong-representability.

then the family is said to be strongly Γ -representable.

If $H \in r^{-1}(\mathbf{x}, q)$ we say that H is related to \mathbf{x} through q . If a system (A, G) belongs to a Γ -representable family of systems and has γ as its associated parameters then (A, G) is said to be γ -representable. If $\gamma_1 = \dots = \gamma_m = 1$ we say that the system is 1-representable. The vector $(K, H, \gamma, l, r, Q, p, c)$ defines the γ -representation and the key parameters are χ_1 and χ_2 .

Comments on Definition 10. In the definition, the hypergraphs H and K could have also been asked to be directed.

By choosing $Q = \{1\}$, $p = 1$ and $\gamma_1 = \dots = \gamma_m = 1$ for all the systems (A, G) , Definition 10 covers the representation notions in [20, 21, 22, 32]. The main purpose of the introduction of the set Q is to accommodate the determinantal condition from [21, Theorem 1]. The different p and γ allow for removing different proportions for different sets $S_i(A, G)$, the projections of the solution set to the coordinates of G^m .

Asking for the bounds on s , h and c to depend on m and on the family of systems as a whole is one of the key points in the representability notion. The existence of r in RP2 and the definition of $r_0(q)$, imply that the labels of the edges of each copy of H in K , ordered by colours, form a solution of the system (A, G) .

For each solution $\mathbf{x} = (x_1, \dots, x_m) \in S(A, G)$, the set Q equipartitions the copies of H in K related to \mathbf{x} . The conditions RP2 and RP3 guarantee, for each \mathbf{x} , q and $i \in [1, m]$, the existence of a set of i -colored edges with size

$$|E_i(\mathbf{x}, q)| = \lambda \gamma_i = c \frac{|K|^s}{|G|} \gamma_i,$$

where c is lower bounded by a function of m such that the following holds. For each edge $e \in E_i(\mathbf{x}, q)$, there are $p \frac{\prod_{j=1}^m \gamma_j}{\gamma_i}$ copies of H in K related to (\mathbf{x}, q) containing e . p is independent on \mathbf{x} , i , q or e . By the existence of r in RP2, any copy of H in K related to \mathbf{x} through q intersects $E_i(\mathbf{x}, q)$ for all $i \in [1, m]$.

If the system is strongly representable, then $E_i(\mathbf{x}, q)$ is the set of edges labelled with $(\mathbf{x})_i$.

In Definition 10, we could have made the constants c to depend on the pair (\mathbf{x}, q) as long as $c_{\mathbf{x}, q} \geq \chi_2$ for any $(\mathbf{x}, q) \in S(A, G) \times Q$. The proof of Theorem 1 in Section 2.2 can be adapted to this case by using the bound χ_2 instead of c .

If the system is γ -strongly-representable, then the new set Q can be considered to be $\{1\}$ at the expense of increasing the value of p to $p|Q|$. Indeed, for any q , the set of hypergraphs H in K related to (\mathbf{x}, q) contains all the edges labelled $(\mathbf{x})_i$. Therefore any edge labelled $(\mathbf{x})_i$ contains $p \frac{\prod_{j=1}^m \gamma_j}{\gamma_i} |Q|$ copies of H related to \mathbf{x} in $\cup_{q \in Q} r^{-1}(\mathbf{x}, q)$.

2.2 Representable systems and the removal lemma

The proof of the removal lemma for representable systems, Theorem 1, uses the coloured version of the hypergraph removal lemma, Theorem 11 in this work. Theorem 11 can be deduced from Austin and Tao's [2, Theorem 1.5]. Alternatively, the coloured version of the hypergraph removal lemma can be proved using the arguments that lead to the colourless version of the hypergraph removal lemma [7, 14, 26, 36], or it can be found in Ishigami's [16].

Theorem 11 (Removal lemma for colored hypergraphs [2]). *For any positive integers r , h , s with $h \geq s \geq 2$ and every $\varepsilon > 0$ there exists $\delta > 0$ depending on r , h , s and ε such that the following holds.*

Let H and K be r -colored s -uniform hypergraphs with $h = |V(H)|$ and $M = |V(K)|$ vertices respectively. If the number of copies of H in K (preserving the colors of the edges) is at most δM^h , then there is a set $E' \subseteq E(K)$ of size at most εM^s such that the hypergraph K' with edge set $E(K) \setminus E'$ is H -free.

Proof of Theorem 1. Let (K, H) be the hypergraph pair that γ -represents the system (A, G) , with $\gamma = (\gamma_1, \dots, \gamma_m)$. Let us denote the labelling by $l : E(K) \rightarrow G$ and the representation function by $r : C(H, K) \rightarrow S(A, G) \times Q$. The components of r are given by $r_0 : C(H, K) \rightarrow S(A, G)$ and $r_q : C(H, K) \rightarrow Q$. Recall that, by Definition 10, if $H_0 = \{e_1, \dots, e_m\}$ is a copy of H in K , then $r_0(H) = (l(e_1), \dots, l(e_m))$. Let K_X be the subhypergraph of K with the same vertex set as K and the edges belonging to $r_0^{-1}(S(A, G, X))$. In other words, $K_X \subset K$ is the hypergraph containing only the edges whose labels belong to the restricted solution set.

By the property RP2 of the γ -representability of the system, the total number of copies of H in K is, for the c and p provided by the representation, at most

$$c \frac{|K|^s}{|G|} p |Q| |S(A, G)| \prod_{i=1}^m \gamma_i.$$

Let $\lambda = c \frac{|K|^s}{|G|}$. Since H has h vertices, it follows that

$$\lambda p |Q| |S(A, G)| \prod_{i=1}^m \gamma_i < |K|^h.$$

On the other hand, the hypothesis $|S(A, G, X)| < \delta |S(A, G)|$, δ to be determined later, implies that the total number of copies of H in K_X is at most

$$\lambda p |Q| |S(A, G, X)| \prod_{i=1}^m \gamma_i < \delta \lambda p |Q| |S(A, G)| \prod_{i=1}^m \gamma_i < \delta |K|^h.$$

We apply the Removal Lemma for colored hypergraphs, Theorem 11, with $\varepsilon' = c\varepsilon/m$. By setting δ according to ε' and H in Theorem 11, we obtain a set of edges $E' \subset E(K_X)$ with cardinality at most $\varepsilon' |K|^s$ such that $K_X \setminus E'$ has no copy of H . We note that δ depends on s, h, m and ε' , which in our context and by the representability, all depend on m and ε .

We next define the sets $X'_i \subset X_i$ as follows. The element x is in X'_i (x is removed from X_i) if E' contains at least $\lambda \gamma_i/m$ edges labelled x and colored i . We observe that

$$|X'_i| \leq \frac{|E'|}{(\lambda \gamma_i/m)} = \frac{m|G|}{c|K|^s} |E'| \frac{1}{\gamma_i} \leq \varepsilon \frac{|G|}{\gamma_i}.$$

We claim that $S(A, G, X \setminus X')$, with $X \setminus X' = \prod_{i=1}^m X_i \setminus X'_i$, is empty. Indeed, pick one element $\mathbf{x} = (x_1, \dots, x_m) \in S(A, G, X)$ and $q \in Q$. By RP2 there are $p\lambda \prod_{i=1}^m \gamma_i$ copies of H in $r^{-1}(\mathbf{x}, q)$. Since $\mathbf{x} \in S(A, G, X)$, all these copies belong to K_X . On the other hand, by RP3, every edge of K coloured i is contained in at most $p \prod_{j \in [1, m] \setminus \{i\}} \gamma_j$ copies of H in $r^{-1}(\mathbf{x}, q)$. Let E'_{i, x_i} denote the set of edges in E' labelled with x_i and colored i . Then

$$\sum_{i=1}^m \left[|E'_{i, x_i}| p \prod_{j \in [1, m] \setminus \{i\}} \gamma_j \right] \geq p\lambda \prod_{j \in [1, m]} \gamma_j$$

as there are no copies related to (\mathbf{x}, q) after E' has been removed. By the pigeonhole principle, at least one of the sets E'_{i, x_i} is such that $|E'_{i, x_i}| > \lambda \gamma_i/m$. By the definition of X'_i , the element x_i belongs to X'_i and thus $\mathbf{x} \notin X \setminus X' \supset S(A, G, X \setminus X')$. This proves the claim and finishes the proof of the result. \square

2.3 Examples of representable systems and removal lemmas

2.3.1 Subhypergraph copies

As expected, the coloured hypergraph removal lemma can be retrieved from Theorem 11. The system of configurations induced by “the copies of an r -coloured k -uniform hypergraph H_0 in an r -coloured k -uniform hypergraph K_0 ” can be represented by Definition 10 as follows. Order the edges of H arbitrarily. $H = H_0$ and $K = K_0$ as the pair of hypergraphs that represents the system. The property A is the map from $E(K)^{|E(H)|}$ to $\{0, 1\}$ such that $A(e_1, \dots, e_{|E(H)|}) = 1$ if and only if the edges $(e_1, \dots, e_{|E(H)|})$ conform a copy of H in K in which e_i is the i -th edge of H with the chosen order. The map l is given by the identity map of the edge in K , r_0 is the identity map induced by the property A , $Q = \{1\}$, $\lambda = c = \gamma_i = 1$. The sets X_i in the removal lemma Theorem 1 are the edges in K_0 coloured using the colour of the i -th edge in H .

2.3.2 Permutations

The copies of $\tau \in \mathcal{S}(t)$ in $\sigma \in \mathcal{S}(n)$, as defined by the set $\Lambda^\tau(\sigma)$ in Section 1.2.5, can be represented using directed and coloured graphs H and K in Definition 10 as follows. Given a finite set V , let $\binom{V}{i}$ denote the set of subsets of

i different elements of V . Let A be the property $A : \binom{[0, n-1]}{2}^{\lfloor \frac{[0, t-1]}{2} \rfloor} \rightarrow \{0, 1\}$ such that $A(e_1, \dots, e_{t(t-1)/2}) = 1$ if and only if the collection of endpoints of the edges $\{e_1, \dots, e_{t(t-1)/2}\}$ configure an m -element set $\{x_0 < \dots < x_{t-1}\}$ in $[0, n-1]$ belonging to $\Lambda^\tau(\sigma)$.

Given a permutation $\sigma \in \mathcal{S}(n)$, let us define the loopless bicoloured directed graph G_σ as follows. The vertex set $V(G_\sigma)$ is given by the n -element set $[0, n-1]$. The directed edge $e = (i, j)$ or $e = \{i \rightarrow j\}$, from i to j , belongs to $E(G_\sigma)$ if and only if $\sigma(i) < \sigma(j)$. The edge $e = \{i \rightarrow j\}$ is painted blue if $i < j$ and painted red if $i > j$. Observe that $|E(G_\sigma)| = \binom{n}{2}$.

We claim that the system for the permutations involved in Proposition 5 is representable with A as above, $H = G_\tau$, $K = G_\sigma$, $m = \binom{t}{2}$, $Q = \{1\}$, $\gamma_i = c = \lambda = 1$ and r_0 given as follows. If $\{x_0 < \dots < x_{t-1}\} \subset [0, n-1] = V(K)$ is a set of indices that generates a copy of H in K , then $r_0(\{x_0 < \dots < x_{t-1}\}) = \{x_0 < \dots < x_{t-1}\}$.

Claim 1. *If H_0 , with $V(H_0) = \{x_0 < \dots < x_{t-1}\}$, is a copy of $G_\tau = ([0, t-1], E(G_\tau))$ in G_σ , then the only map (homomorphism) from $f : V(G_\tau) \rightarrow V(H_0)$ with the property “if $e = \{i \rightarrow j\} \in E(G_\tau)$ and is coloured c , then $\{f(i) \rightarrow f(j)\} \in E(G_\sigma|V(H_0))$ and is coloured c ” is the map $f(i) = x_i$ for all $i \in [0, t-1]$.*

Proof. The map f must be bijective. Indeed, since G_τ is a complete graph if f were not bijective, then the graph induced by $V(H_0)$ would contain a loop as f is a homomorphism, but G_σ is loopless.

If f is not the map $f(i) = x_i$, then there exist a pair $i, j \in [1, t-1]$ with $i < j$ but $f(i) > f(j)$. If the edge between i and j is $e = \{i \rightarrow j\}$, then $f(e) = \{f(i) \rightarrow f(j)\}$. In such case, e is painted blue as $i < j$ and $f(e)$ is painted red as $f(i) > f(j)$, hence f is not an homomorphism. If the edge between i and j is $e' = \{j \rightarrow i\}$, then e is coloured red but $f(e')$ is blue. Therefore, if f is a homomorphism, it has to be the isomorphism with $f(i) = x_i$. \square

Claim 2. *If H_0 , with $V(H_0) = \{x_0 < \dots < x_{t-1}\}$ is a copy of G_τ in G_σ where $x_i \in V(G_\sigma)$ corresponds to the i -th vertex of G_τ , then $\{x_0 < \dots < x_{m-1}\} \in \Lambda^\tau(\sigma)$.*

Proof. Let $e = \{x_i \rightarrow x_j\}$ be an edge in G_σ , then $\sigma(x_i) < \sigma(x_j)$. Since H_0 is a copy of G_τ where x_i corresponds to the i -th vertex of G_τ , $e' = \{i \rightarrow j\}$ is an edge in G_τ meaning that $\tau(i) < \tau(j)$ as wanted. Since the reverse implication also holds, the result is shown. \square

Claim 3. *If $\{x_0 < \dots < x_{t-1}\} \in \Lambda^\tau(\sigma)$ then the graph induced by x_0, x_1, \dots, x_{t-1} in G_σ is a copy of G_τ with the map from $V(G_\tau) = [0, t-1]$ to $\{x_0, x_1, \dots, x_{t-1}\} \subset V(G_\sigma)$ given by $i \mapsto x_i$ for $i \in [0, t-1]$.*

Proof. Assume the pair $\{i, j\} \in \binom{[0, t-1]}{2}$, with $i < j$, is such that $\sigma(x_i) < \sigma(x_j)$. By the construction of G_σ we have the edge $\{x_i \rightarrow x_j\}$ and is painted blue (as $x_i < x_j$). Since $\{x_0 < \dots < x_{t-1}\} \in \Lambda^\tau(\sigma)$, then $\tau(i) < \tau(j)$. Hence G_τ has the edge $\{i \rightarrow j\}$ coloured blue (as $i < j$).

Assume now that the pair $\{i, j\} \in \binom{[0, t-1]}{2}$, with $i < j$, is such that $\sigma(x_i) > \sigma(x_j)$. G_σ contains the edge $\{x_j \rightarrow x_i\}$ painted red (as $x_j > x_i$). On the other side we have $\tau(i) > \tau(j)$ as $\{x_0 < \dots < x_{t-1}\} \in \Lambda^\tau(\sigma)$. Hence G_τ has the edge $\{j \rightarrow i\}$ coloured red (as $j > i$).

Therefore, the map $i \mapsto x_i$, for $i \in [0, t-1]$, is a graph homomorphism preserving the colours and the directions of the edges as claimed. \square

Combining claims 1-3, we observe that r_0 is well defined and the representation of $\Lambda^\tau(\sigma)$ is given by the pair (G_τ, G_σ) with the parameters described above. Proposition 5 is shown by using Theorem 1 with $X_i = \binom{[0, n-1]}{2}$ for all $i \in [t(t-1)/2]$. In this case the proof of Theorem 1 should use, instead of Theorem 11, a removal lemma for directed and coloured graphs that can be obtained by combining the arguments from [1, Lemma 4.1] with [18, Theorem 1.18].¹²

3 Equivalent systems and representability

In this section we assume that the systems are defined by a homomorphism. The definition for μ -equivalent systems is introduced in Section 3.1 and in Section 3.2 the relations between μ -equivalent systems and their representations are explored.

¹²For a detailed argument of how to obtain a removal lemma for directed and coloured graphs, the reader may refer to [41].

3.1 Equivalent systems

Let μ be a positive integer. The homomorphism system (A_2, G_2) with $A_2 : G_2^{m_1} \rightarrow G_2^{k_2}$ is said to be μ -equivalent to the homomorphism system (A_1, G_1) , $A_1 : G_1^{m_1} \rightarrow G_1^{k_2}$, with $m_2 \geq m_1$, if

- $\mu|S(A_1, G_1)| = |S(A_2, G_2)|$.
- There exist an injective map $\sigma : [1, m_1] \rightarrow [1, m_2]$ and affine homomorphisms $\phi_1, \dots, \phi_{m_1}, \phi_i : G_2 \rightarrow G_1$ such that the map

$$\phi(x_1, \dots, x_{m_2}) = (\phi_1(x_{\sigma(1)}), \dots, \phi_{m_1}(x_{\sigma(m_1)}))$$

induces a μ -to-1 surjective map $\phi : S(A_2, G_2) \rightarrow S(A_1, G_1)$.

An affine homomorphism is a map $\phi_i : G_2 \rightarrow G_1$ with $\phi_i(x) = b + \phi'_i(x)$, where ϕ'_i is a homomorphism and b is a fixed element in G_1 . Observe that, if necessary, we can restrict ϕ_i to map from the subgroup $S_i(A_2, G_2)$ (or a coset of the subgroup $S_i((A_2, \mathbf{0}), G_2)$) to $S_i(A_1, G_1)$. If $G_1 = G_2$ and the $\{\phi_i\}_{i \in [1, m_1]}$ are affine automorphisms, then $\phi_i(S_i(A_2, G_2)) = S_i(A_1, G_1)$ and their sizes are the same. In this case the systems are said to be *auto-equivalent*.

3.2 Operations on equivalent systems and representability

The propositions 12 through 15 proved in this section expose how the property of equivalence between systems, as defined in Section 3.1, is related with their representability properties, Definition 10. For this section G , G_1 and G_2 are finite abelian groups and the systems are homomorphism systems.

3.3 1-auto-equivalent systems

Proposition 12 (1-auto-equivalent systems). *Let $((A_2, \mathbf{b}_2), G)$ be a $k_2 \times m_2$ system 1-auto-equivalent to $((A_1, \mathbf{b}_1), G)$, a $k_1 \times m_1$ system. Assume $((A_2, \mathbf{b}_2), G)$ is γ' -representable by (K', H') with constants χ_1, χ_2 . If the edges coloured by $\sigma(1), \dots, \sigma(m_1)$ cover all the vertices of H' , then (A_1, G) is γ -representable with the same constants χ_1, χ_2 and $\gamma_i = \gamma'_{\sigma(i)}$. If $((A_2, \mathbf{b}_2), G)$ is strongly representable, then so is $((A_1, \mathbf{b}_1), G)$.*

Proof of Proposition 12. Let ϕ be the map that defines the 1-auto-equivalence $\phi : S(A_2, G) \rightarrow S(A_1, G)$ with $\phi(x_1, \dots, x_{m_2}) = (\phi_1(x_{\sigma(1)}), \dots, \phi_{m_1}(x_{\sigma(m_1)}))$. Let $(K', H', \gamma', l', r', Q', p', c')$ be the vector defining the γ' -representation for (A_2, G) . Let s be the uniformity of the edges of H' .

The vector $(K, H, \gamma, l, r, Q', p', c')$ defines the γ -representation of $((A_1, \mathbf{b}_1), G)$ as follows. $\gamma_i = \gamma'_{\sigma(i)}$ for $i \in [1, m_1]$. H and K are the hypergraph on the same vertex set of H' and K' respectively, and with the edges given by the colours $\sigma(1), \dots, \sigma(m_1)$. Repaint the edge coloured $\sigma(i)$ with colour i . If $e = \{v_1, \dots, v_s\}$ is an edge coloured $\sigma(i)$ in K' and labelled $l'(e)$, then e is an edge coloured i in K and labelled $l(e) = \phi_i(l'(e))$. $r_q(H_0) = r'_q(H'_0)$ where H'_0 is the unique copy of H' in K' spanned by the vertices of H_0 seen as vertices of K' .

Each copy of H in K induces a unique copy of H' in K' and vice-versa. Moreover, ϕ is a bijection between the solution sets and $(K', H', \gamma', l', r', Q', p', c')$ is a γ' -representation for $((A_2, \mathbf{b}_2), G)$. Therefore, $(K, H, \gamma, l, r, Q, p, c)$ as defined above induces a γ -representation for $((A_1, \mathbf{b}_1), G)$ and have the same constants χ_1 and χ_2 as $(K', H', \gamma', l', r, Q', p', c')$. Since ϕ_i are affine automorphisms, if the representation for $((A_2, \mathbf{b}_2), G)$ is strong, then so is the representation for $((A_1, \mathbf{b}_1), G)$ here presented. \square

3.4 μ -auto-equivalent systems

Proposition 13 (μ -auto-equivalent systems). *Let $((A_2, \mathbf{b}_2), G)$ be a $k_2 \times m_2$ system μ -auto-equivalent to the $k_1 \times m_1$ system $((A_1, \mathbf{b}_1), G)$, $m_2 \geq m_1$. Let*

$$\begin{aligned} \phi : S((A_2, \mathbf{b}_2), G) &\longrightarrow S((A_1, \mathbf{b}_1), G) \\ (x_1, \dots, x_{m_2}) &\longmapsto (x_1, \dots, x_{m_1}) \end{aligned}$$

be the map that defines the μ -auto-equivalence. Assume $((A_2, \mathbf{b}_2), G)$ is $(\gamma'_1, \dots, \gamma'_{m_2})$ -representable by (K', H') with constants χ_1, χ_2 . If the edges coloured by $1, \dots, m_1$ cover all the vertices of H' , then $((A_1, \mathbf{b}_1), G)$ is $(\gamma'_1, \dots, \gamma'_{m_1})$ -representable with χ_1, χ_2 as constants. If $((A_2, \mathbf{b}_2), G)$ is strongly representable, then so is $((A_1, \mathbf{b}_1), G)$.

Proof of Proposition 13. Let ι be a map from $S((A_2, \mathbf{b}_2), G)$ to $[1, \mu]$ where, given $\mathbf{x}_1, \mathbf{x}_2 \in S((A_2, \mathbf{b}_2), G)$ such that $\phi(\mathbf{x}_1) = \phi(\mathbf{x}_2)$ and $\mathbf{x}_1 \neq \mathbf{x}_2$, then $\iota(\mathbf{x}_1) \neq \iota(\mathbf{x}_2)$. If ϕ is a μ -to-1 map, such ι exist, is exhaustive and induces an equipartition in $S((A_2, \mathbf{b}_2), G)$.

Let $(K', H', \gamma', l', r', Q', p', c')$ be the vector defining the γ' -representation for $((A_2, \mathbf{b}_2), G)$. Let s be the uniformity of the edges of H' . The candidate vector $(K, H, \gamma, l, r, Q, p, c)$ is defined as follows.

- $Q = Q' \times [1, \mu]$, $p = p' \prod_{i=m_1+1}^{m_2} \gamma'_i$, $c = c'$, γ is such that $\gamma_i = \gamma'_i$ for $i \in [1, m_1]$.
- H and K are the hypergraphs on the vertex sets of H' and K' respectively. $e = \{v_1, \dots, v_s\}$ is an edge in K coloured $i \in [1, m_1]$ if and only if e is an edge coloured $i \in [1, m_1]$ in K' .
- l is defined by $l(e) = l'(e)$ for e an edge coloured $i \in [1, m_1]$.
- If $H_0 \in C(H, K)$, then $r_q(H_0) = (r'_q(H'_0), \iota(r'_0(H'_0)))$ where H'_0 is the unique copy of H' in K' spanned by the vertices of H_0 , seen as vertices of K' .

Selecting $\mathbf{x} \in S((A_1, \mathbf{b}_1), G)$ and $q = (q', j) \in Q = Q' \times [1, \mu]$ is equivalent to select the $\mathbf{y} \in S((A_2, \mathbf{b}_2), G)$, with $\mathbf{y} \in \phi^{-1}(\mathbf{x})$ such that $\iota(\mathbf{y}) = j$, and $q' \in Q'$, first coordinate of q . Moreover, each copy of H in K induces a unique copy of H' in K' and vice-versa. Therefore, the class of copies of H related to (\mathbf{x}, q) is the same as the copies of H' related to (\mathbf{y}, q') .

Since each edge $e_i \in E(K')$, $i \in [1, m_1]$, is contained in $p' \frac{\prod_{i=1}^{m_2} \gamma'_i}{\gamma'_i}$ copies of H' related to (\mathbf{y}, q') , then it also contains, seen as an edge in K , $p' \frac{\prod_{i=1}^{m_2} \gamma'_i}{\gamma'_i} = p \frac{\prod_{i=1}^{m_1} \gamma_i}{\gamma_i}$ copies of H related to (\mathbf{x}, q) . Therefore, $(K, H, \gamma, l, r, Q, p, c)$ as defined above induces a γ -representation for (A_1, G) and have the same constants χ_1 and χ_2 as $(K', H', \gamma', l', r', Q', p', c')$. Moreover, since ϕ_i is the identity map for each i , if the representation for $((A_2, \mathbf{b}_2), G)$ is strong, then so is the presented representation for $((A_1, \mathbf{b}_1), G)$. \square

3.5 μ -equivalent systems

Proposition 14 (μ -equivalent systems 1). *Let $((A_2, \mathbf{b}_2), G_2)$ be a $k_2 \times m_1$ system μ -equivalent to the $k_1 \times m_1$ system $((A_1, \mathbf{b}_1), G_1)$ with*

$$\begin{aligned} \phi : S((A_2, \mathbf{b}_2), G_2) &\longrightarrow S((A_1, \mathbf{b}_1), G_1) \\ (x_1, \dots, x_{m_1}) &\longmapsto (\phi_1(x_1), \dots, \phi_1(x_{m_1})) \end{aligned}$$

be the map that defines the μ -equivalence. Assume (A_2, G_2) is $(\gamma_1, \dots, \gamma_{m_1})$ -representable by (K', H') with constants χ_1, χ_2 . If $\phi_1 : G_2 \rightarrow G_1$ is surjective and $\phi^{-1}(\mathbf{x}) = \prod_{i=1}^{m_1} \phi_1^{-1}((\mathbf{x})_i)$ then $((A_1, \mathbf{b}_1), G_1)$ is $(\gamma_1, \dots, \gamma_{m_1})$ -representable with the same constants χ_1, χ_2 . If $((A_2, \mathbf{b}_2), G)$ is strongly representable, then so is $((A_1, \mathbf{b}_1), G)$.

Proof of Proposition 14. Observe that, for $i \in [1, m_1]$, $\phi_1(S_i((A_2, \mathbf{b}_2), G_2)) = S_i((A_1, \mathbf{b}_1), G_1)$ as ϕ is surjective. Since ϕ_1 is affine, $|\{y_i \in S_i((A_2, \mathbf{b}_2), G_2) : \phi_1(y_i) = x_i\}|$ is the same for each $x_i \in S_i((A_1, \mathbf{b}_1), G_1)$. Since $\phi^{-1}(\mathbf{x}) = \prod_{i=1}^{m_1} \phi_1^{-1}((\mathbf{x})_i)$ and ϕ_1 is affine, then we can let $\beta = |S_i((A_2, \mathbf{b}_2), G_2)|/|S_i((A_1, \mathbf{b}_1), G_1)|$, as its value is independent of $i \in [1, m_1]$. Therefore, $\mu = \beta^{m_1}$. Additionally, since ϕ_1 is surjective, $\beta = |G_2|/|G_1|$.

Let ι be a map from G_2 to \mathbb{Z}_β such that, if $y_1, y_2 \in G_2$ with $\phi_1(y_1) = \phi_1(y_2)$ and $y_1 \neq y_2$, then $\iota(y_1) \neq \iota(y_2)$. Since ϕ_1 is a β -to-1 map between G_2 and $\phi_1(G_2) = G_1$, then such ι exist, is exhaustive and induces an equipartition of G_2 in β classes. Moreover, ι induces the bijections

$$\begin{aligned} G_2 &\longrightarrow \phi_1(G_2) \times \mathbb{Z}_\beta & \text{and} & & S((A_2, \mathbf{b}_2), G_2) &\longrightarrow S((A_1, \mathbf{b}_1), G_1) \times \mathbb{Z}_\beta^{m_1} \\ y &\longmapsto (\phi_1(y), \iota(y)) & & & \mathbf{y} &\longmapsto (\phi(\mathbf{y}), \iota(\mathbf{y})) \end{aligned}$$

where $\iota((\mathbf{y}_1, \dots, \mathbf{y}_{m_1})) = (\iota(\mathbf{y}_1), \dots, \iota(\mathbf{y}_{m_1}))$. Let $\pi : \mathbb{Z}_\beta^{m_1} \rightarrow \mathbb{Z}_\beta^{m_1}/\langle 1, \dots, 1 \rangle$ be the quotient map.

Let $(K', H', \gamma', l', r', Q', p', c')$ be the vector defining the γ' -representation for $((A_2, \mathbf{b}_2), G_2)$. Let s be the uniformity of the edges of H' . The candidate vector $(K, H, \gamma, l, r, Q, p, c)$ is defined as follows.

- $Q = Q' \times \left[\mathbb{Z}_\beta^{m_1} / \langle (1, \dots, 1) \rangle \right]$, $p = p'$, $c = c'$, $\gamma_i = \gamma'_i$ for $i \in [1, m_1]$.
- H and K are the hypergraphs on the same vertex sets and edge sets as H' and K' respectively. $e = \{v_1, \dots, v_s\}$ is an edge in K coloured $i \in [1, m_1]$ if and only if e is an edge coloured $i \in [1, m_1]$ in K' .
- $l(e) = \phi_1(l'(e))$ if e is an edge coloured $i \in [1, m_1]$ as an edge in K' .
- Given $H_0 \in C(H, K)$, let H'_0 be the unique copy of H' in K' spanned by the vertices of H_0 and let $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_{m_1}) = r'_0(H'_0) \in S((A_2, \mathbf{b}_2), G_2)$ the solution spanned by H'_0 . Then $r_q(H_0) = (r'_q(H'_0), \pi(\iota(\mathbf{y})))$.

Property RP1 is fulfilled with the same parameters and each edge bears a label given by l . The function $r = (r_0, r_q)$ goes from $C(H, K)$ to $S((A_1, \mathbf{b}_1), G_1) \times Q$ by the definition of $r' = (r'_0, r'_q)$, ι , ϕ_1 and π . r is surjective because r' is surjective and $S((A_2, \mathbf{b}_2), G_2)$ is in bijection with $S((A_1, \mathbf{b}_1), G_1) \times \mathbb{Z}_\beta^{m_1}$. Observe that $r^{-1}(\mathbf{x}, q)$ is the union of those $r'^{-1}(\mathbf{y}, q')$, with $\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$, such that $\phi(\mathbf{y}) = \mathbf{x}$ and $q = (q', \pi(\iota(\mathbf{y})))$. This union has β elements, as this is the size of each class in the quotient $\mathbb{Z}_\beta^{m_1} / \langle (1, \dots, 1) \rangle$. Therefore,

$$|r^{-1}(\mathbf{y}, q)| = \beta |r'^{-1}(\mathbf{x}, q')| = \beta p' c' \frac{|K'|^s}{|G_2|} \prod_{i=1}^{m_1} \gamma'_i = p c \frac{|K|^s}{|G_1|} \prod_{i=1}^{m_1} \gamma_i,$$

which shows RP2.

All the solutions $(\mathbf{y}_1, \dots, \mathbf{y}_m) = \mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$ that conform the union just mentioned have the property that any component \mathbf{y}_i takes all the possible β values of $\phi_1^{-1}((\mathbf{x})_i)$. Indeed, from all the solutions $(\mathbf{y}_1, \dots, \mathbf{y}_m) = \mathbf{y}$ that, along with the q' , conform the sets of copies of H given by $r^{-1}(\mathbf{x}, q)$, there is only one solution \mathbf{y} with \mathbf{y}_i having a particular value in $\phi_1^{-1}((\mathbf{x})_i)$. Therefore, if two copies of H in K share an edge $e_i \in H_0 \in r^{-1}(\mathbf{x}, q)$, then they belong to the same set $r'^{-1}(\mathbf{y}, q')$ if seen as copies of H' in K' . Thus, there are $p' \frac{\prod_{j=1}^m \gamma'_j}{\gamma'_i} = p \frac{\prod_{j=1}^m \gamma_j}{\gamma_i}$ copies of H in K sharing e_i . This shows RP3.

Let $\mathbf{x} \in S((A_1, \mathbf{b}_1), G_1)$ and $q = (q', j) \in Q$. Pick e_i with $l(e_i) = (\mathbf{x})_i$ for some $i \in [1, m_1]$. Let $y_i = l'(e_i)$ by seen e_i as an edge in K' . Let \mathbf{y} be the unique solution to $S((A_2, \mathbf{b}_2), G_2)$ such that $\phi(\mathbf{y}) = \mathbf{x}$, $(\mathbf{y})_i = y_i$, and $\pi(\iota(\mathbf{y})) = j$. If $((A_2, \mathbf{b}_2), G_2)$ is strongly representable, there exists a $H'_0, H'_0 \in r'^{-1}(\mathbf{y}, q')$, with $e_i \in H'_0$. If H_0 is the unique copy of H in K on the vertices of H'_0 , then $e_i \in H_0$ and $H_0 \in r^{-1}(\mathbf{x}, q)$. This shows RP4 for the system $((A_1, \mathbf{b}_1), G_1)$ when $((A_2, \mathbf{b}_2), G_2)$ is strongly representable and finishes the proof of the proposition. \square

Proposition 15 (μ -equivalent systems 2). *Let $((A_2, \mathbf{b}_2), G_2)$ be a $k_2 \times m_2$ system μ -equivalent to the $k_1 \times m_1$ system $((A_1, \mathbf{b}_1), G_1)$ with $m_2 \geq m_1$. Let*

$$\begin{aligned} \phi : S((A_2, \mathbf{b}_2), G_2) &\longrightarrow S((A_1, \mathbf{b}_1), G_1) \\ (x_1, \dots, x_{m_2}) &\longmapsto (\phi_1(x_1), \dots, \phi_{m_1}(x_{m_1})) \end{aligned}$$

be the map that defines the μ -equivalence. Assume $((A_2, \mathbf{b}_2), G_2)$ is γ' -strongly-representable by (K', H') with constants χ_1, χ_2 . Assume the following.

- (i) *The edges coloured by $[1, m_1]$ cover all the vertices of H' .*
- (ii) *Given $\mathbf{x} \in S((A_1, \mathbf{b}_1), G_1)$ and $i \in [1, m_1]$, then*

$$|\{\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2) : \phi(\mathbf{y}) = \mathbf{x} \text{ and } (\mathbf{y})_i = y_i\}|$$

is constant for any $y_i \in S_i((A_2, \mathbf{b}_2), G_2)$ with $\phi_i(y_i) = (\mathbf{x})_i$.

Then $((A_1, \mathbf{b}_1), G)$ is strongly γ -representable with $\gamma_i = \gamma'_i \frac{|S_i((A_2, \mathbf{b}_2), G_2)|}{|S_i((A_1, \mathbf{b}_1), G_1)|} \frac{|G_1|}{|G_2|}$ for $i \in [1, m_1]$ and the same constants χ_1, χ_2 .

The condition (ii) is not superfluous. If $A_2 = (1, 1)$, $A_1 = (1, 2)$, $b_1 = b_2 = 0$, $G_1 = G_2 = \mathbb{Z}_2$ and $\phi(x_1, x_2) = (2x_1, x_2)$, then ϕ is one to one but ϕ_1 does not satisfy (ii) for the solution $(0, 0) \in S(A_1, G_1)$. However, the number of solutions $\mathbf{y} \in S(A_2, G_2)$ with $\phi(\mathbf{y}) = \mathbf{x}$ and $(\mathbf{y})_i = y_i$ is either zero (if there is no such solution \mathbf{y} with $(\mathbf{y})_i = y_i$), or it is a positive fixed value for any $i \in [1, m_1]$ if there exist some solution $\mathbf{y} \in \phi^{-1}(\mathbf{x})$ with $(\mathbf{y})_i = y_i$. The reason being that ϕ and ϕ_i are affine homomorphisms and the preimage by ϕ and ϕ_i has a coset/subgroup-like structure. Therefore, the condition (ii) can be rephrased as

- (ii') Given $\mathbf{x} \in S((A_1, \mathbf{b}_1), G_1)$ and $i \in [1, m_1]$ then, for any $y_i \in S_i((A_2, \mathbf{b}_2), G_2)$ with $\phi_i(y_i) = (\mathbf{x})_i$, there exists a $\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$ with $\phi(\mathbf{y}) = \mathbf{x}$ and $(\mathbf{y})_i = y_i$.

Proof of Proposition 15. Since ϕ is surjective, so is $\phi_i : S_i((A_2, \mathbf{b}_2), G_2) \rightarrow S_i((A_1, \mathbf{b}_1), G_1)$ for $i \in [1, m_1]$. Since ϕ_i is affine, $|\{y_i \in S_i((A_2, \mathbf{b}_2), G_2) : \phi_i(y_i) = x_i\}|$ is the same for each $x_i \in S_i((A_1, \mathbf{b}_1), G_1)$. As ϕ and ϕ_i are affine homomorphisms, given $\mathbf{x} \in S((A_1, \mathbf{b}_1), G_1)$, the solutions $\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$ such that $\phi(\mathbf{y}) = \mathbf{x}$ can be partitioned into

$$\{\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2) : \phi(\mathbf{y}) = \mathbf{x}\} = \bigcup_{\substack{y_i \in S_i((A_2, \mathbf{b}_2), G_2) \\ \phi_i(y_i) = (\mathbf{x})_i}} \{\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2) : \phi(\mathbf{y}) = \mathbf{x} \text{ and } (\mathbf{y})_i = y_i\}. \quad (3)$$

By the assumptions, the size of the sets $\{\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2) : \phi(\mathbf{y}) = \mathbf{x} \text{ and } (\mathbf{y})_i = y_i\}$ is independent of each y_i with $\phi_i(y_i) = (\mathbf{x})_i$ and we denote it by μ_i . Therefore (3) is an equipartition. Let $\beta_i = |S_i((A_2, \mathbf{b}_2), G_2)|/|S_i((A_1, \mathbf{b}_1), G_1)|$ be the number of preimages by ϕ_i of each $x_i \in S_i((A_1, \mathbf{b}_1), G_1)$ in $S_i((A_2, \mathbf{b}_2), G_2)$. Then μ_i is such that $\mu_i \beta_i = \mu$.

Let $(K', H', \gamma', l', r', Q', p', c')$ be the vector defining the γ' -strong-representation for $((A_2, \mathbf{b}_2), G_2)$. Let s be the uniformity of the edges of H' . The candidate vector $(K, H, \gamma, l, r, Q, p, c)$ is defined as follows.

- $Q = Q', c = c'$,
- $\gamma_i = \gamma'_i \frac{|S_i((A_2, \mathbf{b}_2), G_2)|}{|S_i((A_1, \mathbf{b}_1), G_1)|} \frac{|G_1|}{|G_2|}$ for $i \in [1, m_1]$. $p = \mu p' \frac{|G_2|^{m_1-1}}{|G_1|^{m_1-1}} \left[\prod_{i=m_1+1}^{m_2} \gamma'_i \right] \left[\prod_{i=1}^{m_1} \frac{|S_i((A_1, \mathbf{b}_1), G_1)|}{|S_i((A_2, \mathbf{b}_2), G_2)|} \right]$.
- H and K are hypergraphs on the same vertex sets as H' and K' respectively. $e = \{v_1, \dots, v_s\}$ is an edge in K (respectively H) coloured $i \in [1, m_1]$ if and only if e is an edge coloured $i \in [1, m_1]$ in K' (respectively H').
- $l(e) = \phi_i(l'(e))$ if e is an edge coloured $i \in [1, m_1]$ as an edge in K' .
- Given $H_0 \in C(H, K)$, let H'_0 be the unique copy of H' in K' spanned by the vertices of H_0 . Then $r_q(H_0) = r'_q(H'_0)$.

RP1 is satisfied for (K, H) with the same bounds and the labelling function l' . By the hypothesis (i), each copy of H' in K' spans a unique copy of H in K and vice-versa. Since

$$r^{-1}(\mathbf{x}, q) = \bigcup_{\mathbf{y} \in \phi^{-1}(S((A_1, \mathbf{b}_1), G_1)) \cap S((A_2, \mathbf{b}_2), G_2)} r'^{-1}(\mathbf{y}, q) \quad (4)$$

and there are μ different $\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$ with $\phi(\mathbf{y}) = \mathbf{x}$, then the union (4) is disjoint and

$$|r^{-1}(\mathbf{x}, q)| = \mu |r'^{-1}(\mathbf{y}, q)| = \mu p' c \frac{|K'|^s}{|G_2|} \prod_{i=1}^{m_2} \gamma'_i.$$

as each set $r'^{-1}(\mathbf{y}, q)$ contains $p' c \frac{|K'|^s}{|G_2|} \prod_{i=1}^{m_2} \gamma'_i$ copies of H' in it.

By the definition of γ and p we have

$$|r^{-1}(\mathbf{x}, q)| = \mu p' c \frac{|K'|^s}{|G_2|} \prod_{i=1}^{m_2} \gamma'_i = \mu p' \frac{|K'|^s}{|G_2|} \left[\prod_{i=m_1+1}^{m_2} \gamma'_i \right] \left[\prod_{i=1}^{m_1} \gamma_i \frac{|S_i((A_1, \mathbf{b}_1), G_1)|}{|S_i((A_2, \mathbf{b}_2), G_2)|} \right] \frac{|G_2|^{m_1}}{|G_1|^{m_1}} = p c \frac{|K|^s}{|G_1|} \prod_{i=1}^{m_1} \gamma_i. \quad (5)$$

Since r' is a γ' -representation function and $\phi = (\phi_1, \dots, \phi_{m_1})$ defines the μ -equivalence between systems (in particular, is surjective), RP2 is satisfied for r .

Given $\mathbf{x} \in S((A_1, \mathbf{b}_1), G_1)$ and $q \in Q$, let e_i be an edge coloured i and with $l(e_i) = (\mathbf{x})_i$. H_0 , a copy of H in K , belongs to $r(\mathbf{x}, q)$ and contains e_i if and only if H_0 , as a copy of H' in K' , contains e_i and belongs to one of the $r'^{-1}(\mathbf{y}, q)$ with $\phi(\mathbf{y}) = \mathbf{x}$. Since $((A_2, \mathbf{b}_2), G_2)$ is γ' -strongly-represented, each set $r'^{-1}(\mathbf{y}, q)$ with $(\mathbf{y})_i = l'(e_i)$

contains an H'_0 , a copy of H' in K' , with $e_i \in H'_0$. By RP3, there are $p' \frac{\prod_{j=1}^{m_2} \gamma'_j}{\gamma'_i}$ copies of H' containing e_i in any set $r'^{-1}(\mathbf{y}, q)$ whenever $(\mathbf{y})_i = l'(e_i)$.

There are $\mu_i = \mu/\beta_i$ solutions $\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$ such that $\phi(\mathbf{y}) = \mathbf{x}$ with $(\mathbf{y})_i = l'(e_i)$. Therefore, there is a total of

$$\begin{aligned} \mu_i p' \frac{\prod_{j=1}^{m_2} \gamma'_j}{\gamma'_i} &= \frac{\mu}{\beta_i} p' \left[\prod_{j=m_1+1}^{m_2} \gamma'_j \right] \frac{\prod_{j=1}^{m_1} \gamma'_j}{\gamma'_i} \\ &= \mu p' \left[\prod_{j=m_1+1}^{m_2} \gamma'_j \right] \frac{\left[\prod_{j=1}^{m_1} \gamma'_j \right] \left[\prod_{j=1}^{m_1} \frac{|S_j((A_2, \mathbf{b}_2), G_2)|}{|S_j((A_1, \mathbf{b}_1), G_1)|} \right] \frac{|G_1|^{m_1}}{|G_2|^{m_1}}}{\gamma'_i \beta_i \frac{|G_1|}{|G_2|}} \frac{|G_2|^{m_1-1}}{|G_1|^{m_1-1}} \prod_{j=1}^{m_1} \frac{|S_j((A_1, \mathbf{b}_1), G_1)|}{|S_j((A_2, \mathbf{b}_2), G_2)|} \\ &= p \frac{\prod_{j=1}^{m_1} \gamma_j}{\gamma_i} \end{aligned}$$

copies of H' in K' through e_i that, seeing as copies of H in K , belong to $r^{-1}(\mathbf{x}, q)$. Hence, the vector $(K, H, \gamma, l, r, Q, p, c)$ fulfills RP3.

To show RP4, choose q and let e_i be an edge in K and let \mathbf{x} be a solution to $((A_1, \mathbf{b}_1), G_1)$ such that $(\mathbf{x})_i = l(e_i)$. By the surjectivity of ϕ there exists a $\mathbf{y} \in S((A_2, \mathbf{b}_2), G_2)$ with $\phi(\mathbf{y}) = \mathbf{x}$. By the assumption (ii), we can choose the solution \mathbf{y} such that $(\mathbf{y})_i = l'(e_i)$. Since $r'^{-1}(\mathbf{y}, q)$ contains a copy H'_0 of H' with $e_i \in H'_0$, then $r^{-1}(\mathbf{x}, q)$ contains H_0 , the copy of H over the vertices of H'_0 , and satisfies $e_i \in H_0$. This shows RP4 and finishes the proof of Proposition 15. \square

Comment. In Propositions 13, 14, and 15, the permutation σ has been omitted as the variables are assumed to be properly ordered so that $\sigma(i) = i$ for $i \in [1, m_1]$.

4 Proof of Theorem 2: from G to \mathbb{Z}_n^t

Sketch of the proof of Theorem 2. Let $((A, \mathbf{b}), G)$ be a homomorphism system with $A : G^m \rightarrow G^k$. We will see that each element of the family of the homomorphism systems on m variables and k equations, with $m \geq k + 2$, admits a representation where the constants χ_1 and χ_2 involved only depend on m . For any given system $((A, \mathbf{b}), G)$, we find a sequence of μ -equivalent systems $\{((A^{(i)}, \mathbf{b}^{(i)}), G^{(i)})\}$, $i \in [1, \kappa]$ for some $\kappa \in \mathbb{N}$, such that $((A^{(\kappa)}, \mathbf{b}^{(\kappa)}), G^{(\kappa)})$ is strongly representable. Moreover, the sequence is equipped with affine morphisms $\phi : S((A^{(i+1)}, \mathbf{b}^{(i+1)}), G^{(i+1)}) \rightarrow S((A^{(i)}, \mathbf{b}^{(i)}), G^{(i)})$ that fulfill the hypotheses of an appropriate proposition from Section 3.2. By concatenating these propositions, we obtain the final result Proposition 33. The final argument of the construction is summarized in Section 5.10. For the cases regarding $m < k + 2$ and to show the second part of Theorem 2, the additional argument from Section 6 is used.

The sequence of systems $\{((A^{(i)}, \mathbf{b}^{(i)}), G^{(i)})\}_{i \in [1, \kappa]}$ deals with different features of the solution set $S((A, \mathbf{b}), G)$ so that, for the last element of the sequence, a 1-strong-representation can be found using the methods from [21]. In Section 4.1 the case of non-homogeneous systems is reduced to the homogeneous case (A, G) . In Section 4.2, we observe that the representation for any abelian group can be reduced to the homocyclic case \mathbb{Z}_n^t , for some appropriate t and n . Section 5 is devoted to the γ -representation for any system with $G = \mathbb{Z}_n^t$. In Section 5.1 we describe the interpretation of A as an integer matrix in the case of $G = \mathbb{Z}_n^t$. Once we have an integer matrix, we prepare the system for any determinantal in Section 5.2 while Section 5.5 prepares the systems to deal with the cases where $\gamma \neq 1$. Sections 5.7 and Section 5.8 are devoted to the representation by hypergraphs using the tools detailed in Section 5.6.

4.1 Representation and the independent vector

Proposition 16 below shows that we can restrict ourselves to consider homogeneous systems $A\mathbf{x} = 0$.

Proposition 16 (Representation: any independent vector). *Either there is no solution to $A\mathbf{x} = \mathbf{b}$, $x \in G^m$ or $((A, 0), G)$ is 1-auto-equivalent to $((A, \mathbf{b}), G)$.*

Proof of Proposition 16. Assume that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_m)$. The map $\phi : S((A, \mathbf{0}), G) \rightarrow S((A, \mathbf{b}), G)$ with $\phi((\mathbf{x}_1, \dots, \mathbf{x}_m)) = (\mathbf{x}_1 + \mathbf{y}_1, \dots, \mathbf{x}_m + \mathbf{y}_m)$ defines a 1-auto-equivalence. \square

4.2 Representability for \mathbb{Z}_n^t implies representability for G

This section shows how to obtain, for some system A' and integers t and n , a system (A', \mathbb{Z}_n^t) μ -equivalent to the given homogeneous system (A, G) . Moreover, the map defining the μ -equivalence fulfills the hypothesis of Proposition 14. Thus, a representation result for any system (A, \mathbb{Z}_n^t) is enough.

By the Fundamental Theorem of Finite Abelian Groups, G can be expressed, for some $n_1, \dots, n_t > 1$ as the product of cyclic groups $G = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_t}$, with $n_i | n_j$ for $i \geq j$. Let $G' = \mathbb{Z}_{n_1}^t$. The group G can be seen as a quotient of G' . Let us denote by $\tau : G' \rightarrow G$ the quotient map

$$\tau(a_1, \dots, a_t) = \left(\frac{n_1}{n_1} a_1, \dots, \frac{n_1}{n_t} a_t \right)$$

and let $\beta = |G'|/|G|$. Let τ' denote the extension of τ from G' to G'^m using the diagonal action; if $(x_1, \dots, x_m) = \mathbf{x} \in G'^m$ then $\tau'(\mathbf{x}) = (\tau(x_1), \dots, \tau(x_m))$.

Recall that the set of homomorphisms $A : G^m \rightarrow G^k$ are in bijection with $k \times m$ homomorphism matrices $(\vartheta_{i,j})$ for some homomorphisms $\vartheta_{i,j} : G \rightarrow G$ with

$$\begin{pmatrix} \vartheta_{1,1} & \cdots & \vartheta_{1,m} \\ \vdots & \ddots & \vdots \\ \vartheta_{k,1} & \cdots & \vartheta_{k,m} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \iff \sum_{i=1}^m \vartheta_{j,i}(x_i) = b_j, \forall j \in [1, k].$$

See, for instance [39, Section 13.10, p. 66].

By considering the matrix of homomorphisms $(\vartheta'_{i,j}) = (\vartheta_{i,j} \circ \tau)$, any homomorphism $A : G^m \rightarrow G^k$ induces a homomorphism $A' : G'^m \rightarrow G'^k$. If we see $\mathbf{b} \in G'^k \supset G^k$, then the system $((A, \mathbf{b}), G)$ induces a system $((A', \mathbf{b}), G')$. Indeed if $\mathbf{y} \in S((A', \mathbf{b}), G')$ then $\tau'(\mathbf{y}) \in S((A, \mathbf{b}), G)$ and for any $\mathbf{x} \in S((A, \mathbf{b}), G)$, then $\tau'^{-1}(\mathbf{x}) \subset S((A', \mathbf{b}), G')$ and $\tau'^{-1}(\mathbf{x}) \neq \emptyset$.

Observation 17. $\tau' : S((A', \mathbf{b}), G') \rightarrow S((A, \mathbf{b}), G)$ is surjective. If $S_i((A, \mathbf{b}), G)$ is the translated subgroup obtained by projecting the solution set to the i -th coordinate of G^m , then $\tau^{-1}(S_i((A, \mathbf{b}), G)) = S_i((A', \mathbf{b}), G')$ and

$$\frac{|G|}{|S_i((A, \mathbf{b}), G)|} = \frac{|G'|}{|S_i((A', \mathbf{b}), G')|}.$$

Moreover, for any $\mathbf{x} \in S((A, \mathbf{b}), G)$, $\tau'^{-1}(\mathbf{x}) = \prod_{i=1}^m \tau^{-1}((\mathbf{x})_i)$.

Remark 18. Observe that $((A', \mathbf{b}), \mathbb{Z}_n^t)$ is μ -equivalent to $((A, \mathbf{b}), G)$ with the surjective map $\tau' : S(A', G') \rightarrow S(A, G)$ and that the hypotheses of Proposition 14 regarding the map $\phi = \tau'$ hold by Observation 17.

Therefore, using Proposition 16, it is enough to find a γ -representation for $((A', \mathbf{0}), \mathbb{Z}_n^t)$, alternatively denoted by (A', \mathbb{Z}_n^t) , with $\gamma_i = |\mathbb{Z}_n^t|/|S_i(A', \mathbb{Z}_n^t)|$.

5 Proof of Theorem 2: γ -representability of (A, \mathbb{Z}_n^t)

In this section we prove the γ -representability of (A, \mathbb{Z}_n^t) for homomorphism systems A with $m \geq k + 2$. The other cases with $m < k + 2$ are treated in Section 6. Following Section 4.2, A can be seen as a $k \times m$ matrix of homomorphisms. As previously mentioned, the construction involves creating a sequence of μ -equivalent sequence, each element of the sequence being a modification of the pair matrix-group from the previous one.

5.1 From a homomorphism to an integer matrix

Let $g_i = (0, \dots, 0, \overset{i}{1}, 0, \dots, 0)$, $i \in [1, t]$, be the canonical generators of $G = \mathbb{Z}_n^t$. Any variable x_i in \mathbb{Z}_n^t can be decomposed into t variables $x_i = (x_{i,1}, \dots, x_{i,t})$, with $x_{i,j} \in \mathbb{Z}_n$. Therefore, any $k \times m$ homomorphism matrix in

\mathbb{Z}_n^t can be expressed as a $tk \times tm$ integer matrix by replacing each homomorphism $\psi : \mathbb{Z}_n^t \rightarrow \mathbb{Z}_n^t$ by a $t \times t$ integer matrix $\Psi = (\psi_{i,j})$; $\psi_{i,j}$ is the coefficient of g_i in the image of g_j by ψ expressed as a linear combination of the generators g_1, \dots, g_t . Indeed, the image of g_j by ψ is an element of \mathbb{Z}_n^t , hence it can be thought of as a tuple in $[0, n-1]^t$; $\psi_{i,j}$ is the i -th component of such tuple.

With these considerations, the system A can be interpreted as an integer system of dimensions $tk \times tm$ with the variables in \mathbb{Z}_n : $A(x_{1,1}, \dots, x_{1,t}, \dots, x_{m,1}, \dots, x_{m,t})^\top = 0$ with

$$A = \begin{pmatrix} \Psi_{1,1} & \cdots & \Psi_{1,m} \\ \vdots & \ddots & \vdots \\ \Psi_{k,1} & \cdots & \Psi_{k,m} \end{pmatrix}, \text{ where } \Psi_{i,j} \text{ is a } t \times t \text{ block of integers.}$$

If A^i is a column of zeros in A , we can exchange it with any column vector whose components are multiples of n . If the determinant of the $kt \times kt$ submatrix of A formed by the first kt columns is zero, as a matrix with coefficients in \mathbb{Z} , then we add appropriate multiples of n to the main diagonal so that the modified matrix has non-zero determinant in \mathbb{Z} .¹³ The modified matrix and the original are equivalent in \mathbb{Z}_n .

Even though A is treated as an integer matrix for most of Section 5, the arguments should take in consideration the origins of A as a homomorphism matrix. In particular, the t variables $x_{i,1}, \dots, x_{i,t}$ coming from x_i are kept consecutive as they represent a unique variable x_i .

5.2 Union of systems: independent vectors simulation

Let $S(A)$ denote the Smith Normal Form of A . Recall that the i -th determinantal divisor of A , denoted by $D_i(A)$ and named i -th determinantal for short, is the greatest common divisor of the determinants of all the $i \times i$ submatrices of A (choosing i rows and i columns). The product of the first i elements in the diagonal of $S(A)$, $\prod_{j=1}^i d_j$, equals the i -th determinantal of A , so $D_i(A) = \prod_{j=1}^i d_j$. A_i denotes the i -th row of A while A^i its i -th column.

Proposition 19 (Row multiples). *Let A be a $k \times m$, $m \geq k$ integer matrix. Let d_1, \dots, d_k denote its diagonal elements of the Smith Normal Form of A . There is a matrix $A^{(1)}$, equivalent to A (row reduced), such that the row $A_j^{(1)}$ satisfies*

$$\gcd(\{A_{j,i}^{(1)}\}_{i \in [1,m]}) = d_j.$$

Furthermore, assume that $d_i \neq 0$ for $i \in [1, k]$. The matrix $A^{(2)}$, obtained from $A^{(1)}$ by dividing the row $A_j^{(1)}$ by d_j , has k -th determinantal one.

Proof of Proposition 19. Let $S = U^{-1}AV^{-1}$ be the Smith Normal Form of A , where U and V are integer unimodular matrices that convey, respectively, the row and column operations that transform A into S . We have $S = (D|0)$, where D is a $k \times k$ diagonal integer matrix with $\det(D) = D_k(A)$ and 0 is an all-zero $k \times (m-k)$ matrix. d_i is the i -th element in the main diagonal of D . Let $A^{(1)} = U^{-1}A = SV$. Notice that the system $A^{(1)}\mathbf{x} = \mathbf{0}$ is equivalent to $A\mathbf{x} = \mathbf{0}$.

As $A^{(1)}$ has been obtained from S by column operations using integer coefficients, the j -th row $A_j^{(1)}$ is formed by integer multiples of d_j . Since V is unimodular, then

$$\gcd(\{A_{j,i}^{(1)}\}_{i \in [1,m]}) = d_j,$$

which proves the first part of the statement. Let $A^{(2)}$ be the matrix obtained by dividing each row $A_j^{(1)}$ by d_j . We have $A^{(2)} = S^{(2)}V$, where $S^{(2)} = (I_k|0)$ is the Smith Normal Form of $A^{(2)}$ and I_k is the $k \times k$ identity matrix. This completes the proof. \square

The integer d_i induces a homomorphism $d_i : G \rightarrow G$ with $d_i(x) = d_i x = \sum_{j=1}^{d_i} x$. Let $\mathcal{P}_{d_i}(G)$ denote the set $d_i^{-1}(0) \subset G$, this is, the subgroup of preimages of 0 by the homomorphism induced by d_i inside G .

¹³This can be done as, for instance, n^{ikt} grows faster than $(kt)!n^i$ when i increases and $n \geq 2$.

Observation 20 (Solution set). *Using Proposition 19:*

$$S(A, G) = \bigcup_{\mathbf{b}} S((A^{(2)}, \mathbf{b}), G), \text{ for } \mathbf{b} \in \prod_{i=1}^k \mathcal{P}_{d_i}(G),$$

where d_i is the greatest common divisor of the i -th row of $A^{(1)}$.

Proof of Observation 20. Let $\mathbf{x} \in G^m$ be a solution to $A\mathbf{x} = \mathbf{0}$, or, equivalently, $A^{(1)}\mathbf{x} = \mathbf{0}$. Observe the j -th equation for $A^{(1)}$:

$$A_{j,1}^{(1)}x_1 + \cdots + A_{j,m}^{(1)}x_m = 0 \iff d_j (A_{j,1}^{(2)}x_1 + \cdots + A_{j,m}^{(2)}x_m) = 0.$$

Thus, $A_{j,1}^{(2)}x_1 + \cdots + A_{j,m}^{(2)}x_m$ is an element of $\mathcal{P}_{d_j}(G)$. Doing the same for all the rows (equations) of the system gives us that $A^{(2)}\mathbf{x} = \mathbf{b}$ for some independent vector \mathbf{b} in $\prod_{i=1}^k \mathcal{P}_{d_i}(G) \subset G^k$. Also, any solution to $A^{(2)}\mathbf{x} = \mathbf{b}$ for some $\mathbf{b} \in \prod_{i=1}^k \mathcal{P}_{d_i}(G)$ is a solution to $A^{(1)}\mathbf{x} = \mathbf{0}$ by multiplying the i -th equation by d_i . \square

We introduce dummy variables $y_j \in G$ to account for those independent vectors that occur by Observation 20. The variables $y_i \in G$ are called *simulating* variables.

Observation 21 (Simulating the independent vector for \mathbb{Z}_n^s). *Assume $G = \mathbb{Z}_n^s$. For each row $A_j^{(1)}$, the equation*

$$A_{j,1}^{(2)}x_1 + \cdots + A_{j,m}^{(2)}x_m - \frac{n}{\gcd(n, d_j)}y_j = 0,$$

where y_j is a new variable with $y_j \in G$, is $|\mathcal{P}_{n/\gcd(n, d_j)}(G)|$ -auto-equivalent to

$$A_{j,1}^{(1)}x_1 + \cdots + A_{j,m}^{(1)}x_m = d_j (A_{j,1}^{(2)}x_1 + \cdots + A_{j,m}^{(2)}x_m) = 0.$$

The application $(x_1, \dots, x_m, y_i) \rightarrow (x_1, \dots, x_m)$ gives the $|\mathcal{P}_{n/\gcd(n, d_j)}(G)|$ -auto-equivalence.

Moreover, for each value of the j -th component of the independent vector $g \in \mathcal{P}_{d_j}(G)$, there are $|G|/|\mathcal{P}_{d_j}(G)|$ values for y_j with

$$\frac{n}{\gcd(n, d_j)}y_j = g.$$

Proof of Observation 21. Since $G = \mathbb{Z}_n^s$ then $\mathcal{P}_{d_j}(G) \cong \mathbb{Z}_{\gcd(n, d_j)}^s$. Observe that the introduction of \bar{y}_j in

$$A_{j,1}^{(2)}x_1 + \cdots + A_{j,m}^{(2)}x_m - \bar{y}_j = 0$$

with $\bar{y}_j \in \mathbb{Z}_{\gcd(n, d_j)}^s$ simulates the independent vector.

As $\frac{n}{\gcd(n, d_j)} : \mathbb{Z}_n^s \rightarrow \mathbb{Z}_{\gcd(n, d_j)}^s$ with $\frac{n}{\gcd(n, d_j)}(g) = \frac{n}{\gcd(n, d_j)}g$ is a $|\mathcal{P}_{n/\gcd(n, d_j)}(G)|$ -to-1 surjective homomorphism, we can replace the variable $\bar{y}_j \in \mathbb{Z}_{\gcd(n, d_j)}^s$ by the variable $y_j \in \mathbb{Z}_n^s$ multiplied by $\frac{n}{\gcd(n, d_j)}$ and obtain the two parts of the observation. \square

Let $A^{(3)}$ denote the new matrix of the system with the simulating variables. This is, $A^{(3)} = (A^{(2)} Y)$ where Y is a collection of columns of a $k \times k$ diagonal integer matrix.

Remark. If A is a $tk \times tm$ integer matrix coming from a homomorphism matrix, then we use Observation 21 on each row with $G = \mathbb{Z}_n$ (or $s = 1$). Additionally, Observation 20 should consider the matrices as $tk \times tm$ integer matrices and $\mathbf{b} \in \mathbb{Z}_n^{tk}$. Adding the simulating variables is only needed when $\gcd(d_i, n) \neq 1$. To simplify the arguments, we may add some additional columns in the matrix Y , with its coefficients being multiples of n , so that the final matrix $A^{(3)}$ has dimensions $tk \times tm^{(3)}$, with $m^{(3)} = m + k$. Since $D_{tk}(A^{(2)}) = 1$, then $D_{tk}(A^{(3)}) = 1$.

Remark 22. The system $(A^{(3)}, \mathbb{Z}_n^t)$ is μ -auto-equivalent to $(A^{(1)}, \mathbb{Z}_n^t)$ with

$$\begin{aligned} \phi : S(A^{(3)}, \mathbb{Z}_n^t) &\longrightarrow S(A^{(1)}, \mathbb{Z}_n^t) \\ (x_1, \dots, x_m, x_{m+1}, \dots, x_{m^{(3)}}) &\longmapsto (x_1, \dots, x_m), \end{aligned}$$

where μ is the number of preimages by ϕ of each $\mathbf{x} \in S(A^{(1)}, \mathbb{Z}_n^t)$ in $S(A^{(3)}, \mathbb{Z}_n^t)$. If $m^{(3)} = m + k$ then $\mu = \prod_{i=1}^{tk} \frac{n}{\gcd(d_i, n)}$.

5.3 From the determinantal to the determinant

Lemma 23 (Matrix extension, Lemma 9 in [21]). *Let A be a $k \times m$ integer matrix, $m \geq k$. There is an $m \times m$ integer matrix N that contains A in its first k rows and is such that $\det(N) = D_k(A)$.*

Let us include a proof for completeness.

Proof of Lemma 23. Let $S = UAV = (D|0)$ be the Smith Normal Form of A , where U and V are unimodular matrices and D is a $k \times k$ diagonal matrix. Consider

$$S' = \begin{pmatrix} D & 0 \\ 0 & I_{m-k} \end{pmatrix} \quad \text{and} \quad U' = \begin{pmatrix} U & 0 \\ 0 & I_{m-k} \end{pmatrix}.$$

Then $N = U'^{-1}S'V^{-1}$ is an integer matrix as U' is unimodular and satisfy the thesis of the lemma. \square

As $D_{tk}(A^{(2)}) = 1$, we use Lemma 23 to extend the $tk \times tm$ integer matrix $A^{(2)}$ to a $tm \times tm$ determinant 1 integer matrix

$$N = \begin{pmatrix} A^{(2)} \\ M \end{pmatrix}, \quad \det \begin{pmatrix} A^{(2)} \\ M \end{pmatrix} = \det(N) = 1,$$

which is a part of the matrix

$$A^{(4)} = \begin{pmatrix} A^{(2)} & 0 & Y \\ M & I_{tm-tk} & 0 \end{pmatrix}.$$

Therefore, the matrix $A^{(4)}$ can be row reduced into a new matrix $A^{(5)}$ in such a way that

$$A^{(5)} = (I_{tm} \ B) \sim A^{(4)}$$

for some $tm \times [(tm - tk) + (tm^{(3)} - tm)] = tm \times t\overline{m}^{(4)}$ integer matrix B . Moreover, we can assume that the columns of the matrix I_{tm} from $A^{(5)}$ correspond to the ordered original variables $((x_{1,1}, \dots, x_{1,t}), \dots, (x_{m,1}, \dots, x_{m,t}))$. Observe that $A^{(5)}$ has tm rows and $tm^{(5)}$ columns, where $m^{(5)} = m + \overline{m}^{(4)}$.

Remark 24. *The system $((A^{(4)}, \mathbf{0}), G)$, hence $((A^{(5)}, \mathbf{0}), G)$, is 1-auto-equivalent to $((A^{(3)}, \mathbf{0}), G)$. Indeed, for any solution $\mathbf{y} \in S(A^{(4)}, G)$ there exists one, and only one, solution $\mathbf{x} \in S(A^{(3)}, G)$ such that the projection*

$$\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_{tm^{(5)}}) \mapsto (\mathbf{y}_1, \dots, \mathbf{y}_{tm}, \mathbf{y}_{tm+(tm-tk)+1}, \dots, \mathbf{y}_{tm^{(5)}})$$

*gives \mathbf{x} .*¹⁴

Let us show an observation that is helpful in Section 5.8.

Observation 25. *Let $A = (A' \ B)$, with B being a $k \times m$, $m \geq k$ integer matrix and A' denotes a square matrix of dimension k . Let n be a positive integer and assume that $\gcd(D_k(B), n) = 1$. Then, for any value of x_1, \dots, x_k , $x_i \in \mathbb{Z}_n$, there are n^{m-k} values for $(x_{k+1}, \dots, x_{k+m}) \in \mathbb{Z}_n^m$ with $A\mathbf{x} = 0$.*

Proof of Observation 25. Extend the matrix A with Lemma 23 to a 1-auto-equivalent system

$$A' = \begin{pmatrix} A' & B & 0 \\ 0 & M & I_{m-k} \end{pmatrix} \quad \text{with} \quad \gcd \left(\det \begin{pmatrix} B \\ M \end{pmatrix}, n \right) = 1.$$

Select a value for x_1, \dots, x_k and any value for the last $m - k$ variables of A' . Then the value of the variables x_{k+1}, \dots, x_{k+m} in \mathbb{Z}_n is uniquely determined as the determinant is coprime with n . \square

¹⁴The coordinates to be omitted correspond to the columns of $(0 \ I_{tm-tk})^\top$ for $A^{(4)}$. The value of these variables is determined by the values on the first m coordinates.

5.4 Grouping the variables: on the matrix B

In Section 5.1 we have assigned an integer matrix in \mathbb{Z}_n to a given homomorphism matrix. Let us partially reverse this transformation. Consider $A^{(5)}$ to be formed by $mm^{(5)}$ blocks of size $t \times t$, where $m^{(5)} = \overline{m}^{(4)} + m$. Let $\mathcal{A}_i^{(5)}$ be the matrix formed by the i -th row of blocks. Omitting the blocks of zeroes from the I_{tm} part of $A^{(5)}$, $\mathcal{A}_i^{(5)}$ can be written as

$$\mathcal{A}_i^{(5)} = (I_t \quad \mathcal{B}_i),$$

where \mathcal{B}_i corresponds to the rows $B_{(i-1)t+1}, \dots, B_{(i-1)t+t}$ from $A^{(5)}$. We can assume that $D_t(\mathcal{B}_i) \neq 0$ in \mathbb{Z} . Otherwise, we can add an appropriate multiple of n to each of the elements of \mathcal{B}_i ; the new matrix is equivalent in \mathbb{Z}_n and has non-zero determinantal in \mathbb{Z} .

By Proposition 19, \mathcal{B}_i has an equivalent, row reduced, matrix $\mathcal{B}_i^{(1)}$ where the greatest common divisors of the rows are the elements in the diagonal of the Smith Normal Form of \mathcal{B}_i . By performing such row reductions into $\mathcal{A}_i^{(5)}$, or in the whole $A^{(5)}$ using the corresponding rows, the matrix I_t turns into a unimodular matrix U_i related with the row operations conducted on \mathcal{B}_i to obtain $\mathcal{B}_i^{(1)}$. Since $D_t(\mathcal{B}_i^{(1)}) = D_t(\mathcal{B}_i) \neq 0$, $\mathcal{B}_i^{(1)}$ has no zero row in \mathbb{Z} .

As U_i is unimodular, it induces an automorphism in $G = \mathbb{Z}_n^t$ denoted by $\phi_i^{(1)} : G \rightarrow G$, with

$$\phi_i^{(1)}(x) = \phi_i^{(1)}((x_1, \dots, x_t)) = \left[U_i^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_t \end{pmatrix} \right]^\top.$$

Consider the matrix $A^{(6)} = (I_{tm} \quad B^{(1)})$ where $B^{(1)}$ is formed by collecting all the rows from $\mathcal{B}_i^{(1)}$, $i \in [1, m]$.

Remark 26. $(A^{(6)}, \mathbb{Z}_n^t)$ is 1-auto-equivalent to $(A^{(5)}, \mathbb{Z}_n^t)$ with

$$\begin{aligned} \phi : S(A^{(6)}, \mathbb{Z}_n^t) &\longrightarrow S(A^{(5)}, \mathbb{Z}_n^t) \\ \mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_{m^{(5)}}) &\longmapsto (\phi_1^{(1)}(\mathbf{x}_1), \dots, \phi_{m^{(5)}}^{(1)}(\mathbf{x}_{m^{(5)}})) \end{aligned}$$

being the map between the solutions sets.

5.5 Towards $\gamma \neq 1$: constructing several systems

We create several auxiliary systems to achieve an appropriate $\gamma \neq 1$ that are combined in Section 5.8. The purpose of its combination is to create a strongly 1-representable system $(A^{(7)}, G^{(7)})$ with $S_i(A^{(7)}, G^{(7)}) = G^{(7)}$ for any i . $(A^{(7)}, G^{(7)})$ is μ -equivalent to $(A^{(6)}, \mathbb{Z}_n^t)$ and the map of the μ -equivalence fulfills the hypotheses of Proposition 15. See Remark 31.

Let $\mathcal{B}_i^{(2)}$ be the matrix obtained from $\mathcal{B}_i^{(1)}$ by dividing each row of $\mathcal{B}_i^{(1)}$, denoted by $\mathcal{B}_{i,[j]}^{(1)}$ with $j \in [1, t]$, by $d_{i,j} = \gcd(\mathcal{B}_{i,[j]}^{(1)})$. Therefore, the greatest common divisor of each row in $\mathcal{B}_i^{(2)}$ is one.¹⁵ Let $B^{(2)}$ be the matrix formed by collecting the rows in $\mathcal{B}_i^{(2)}$, $i \in [1, m]$. That is to say, for $i \in [1, m]$ and $j \in [1, t]$, the $(i-1)t + j$ -th row of $B^{(2)}$ is the j -th row of $\mathcal{B}_i^{(2)}$.

Given $i \in [1, m]$ and $j \in [1, t]$, let $\mathcal{B}_{i(j)}^{(2)}$ denote the matrix

$$\mathcal{B}_{i(j)}^{(2)} = \begin{pmatrix} B_{[t(i-1)+1, t(i-1)+j-1]}^{(2)} \\ \mathcal{B}_{i,[j]}^{(1)} \\ B_{[t(i-1)+j+1, ti]}^{(2)} \end{pmatrix}$$

where $B_{[i_1, i_2]}^{(2)}$ denotes the set of rows with indices in $[i_1, i_2]$ from $B^{(2)}$ and $\mathcal{B}_{i,[j]}^{(1)}$ denotes the j -th row of $\mathcal{B}_i^{(1)}$. This is, all the rows of $\mathcal{B}_{i(j)}^{(2)}$ are the same as the rows of $\mathcal{B}_i^{(2)}$ except the j -th, which is the same as the j -th row in $\mathcal{B}_i^{(1)}$.

¹⁵We could have chosen to divide the coefficients of the row $\mathcal{B}_{i,j}^{(1)}$ by the minimum $\overline{d}_{i,j}$ such that $\gcd(d_{i,j}/\overline{d}_{i,j}, n) = 1$.

For $i \in [1, m]$ and $j \in [1, t]$ let $J_{(i,j)}$ be the matrix formed by

$$\begin{aligned} J'_{(i,j)} &= \begin{pmatrix} I_{t(i-1)} & 0 & 0 & 0 & 0 & B_{[1,t(i-1)]}^{(2)} & 0 & 0 \\ 0 & I_t & 0 & e_j & 0 & \mathcal{B}_{i(j)}^{(2)} & 0 & 0 \\ 0 & 0 & I_{t(m-i)} & 0 & 0 & B_{[ti+1,tm]}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{t-1} & 0 & 0 & I_{t-1} \end{pmatrix} \\ &\sim \begin{pmatrix} I_{t(i-1)} & 0 & 0 & 0 & 0 & B_{[1,t(i-1)]}^{(2)} & 0 & 0 \\ 0 & I_t & 0 & 0 & 0 & \mathcal{B}_{i(j)}^{(2)} & -e_j & 0 \\ 0 & 0 & I_{t(m-i)} & 0 & 0 & B_{[ti+1,tm]}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{t-1} & 0 & 0 & I_{t-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{t(m+1)} & B_{(i,j)}^{(3)} \end{pmatrix} = J_{(i,j)} \end{aligned}$$

where $e_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)^\top \in \mathbb{Z}^t$.

The variables in the system associated to $J_{(i,j)}$ take values over $G_{(i,j)} = d_{i,j}^{-1}(0) \subset \mathbb{Z}_n$, the subgroup of \mathbb{Z}_n formed by the preimage of zero by the homomorphism induced by $d_{i,j}$ in \mathbb{Z}_n .¹⁶ The matrix $J_{(i,j)}$ can be considered as a homomorphisms system over $G_{(i,j)}$ or over $G_{(i,j)}^t$, by considering $J_{(i,j)}$ to be formed by blocks of size $t \times t$. In the first case the system is denoted by $(J_{(i,j)}, G_{(i,j)})$ and, in the second, by $(J_{(i,j)}, G_{(i,j)}^t)$.

With respect to $A^{(6)}$, t equations and $2t$ variables in $G_{(i,j)}$ have been added to $J_{(i,j)}$. The added variables form the $(m+1)$ -th block of t variables in $J_{(i,j)}$ and the last block of t variables over $G_{(i,j)}$. The added equation corresponds to the last one in $J_{(i,j)}$ and it involves the $2t$ variables added.

Let $J_{(1,0)}$ be the system induced by the matrix

$$J_{(1,0)} = \begin{pmatrix} I_{tm} & 0 & B^{(2)} & 0 \\ 0 & I_t & 0 & I_t \end{pmatrix} = \begin{pmatrix} I_{t(m+1)} & B_{(1,0)}^{(3)} \end{pmatrix}$$

that configures a system $(J_{(1,0)}, \mathbb{Z}_n)$ or $(J_{(1,0)}, \mathbb{Z}_n^t)$ if $J_{(1,0)}$ is seen as a block matrix. Let us denote $G_{(1,0)} = \mathbb{Z}_n$.

Let $\Upsilon = (1, 0) \cup \{[1, m] \times [1, t]\}$. The systems J_κ , $\kappa \in \Upsilon$, thought of as integer matrices, have some common properties.

- (i) J_κ have $m^{(j)} = m^{(6)} + 2 = m^{(5)} + 2$ variables and $k^{(j)} = k^{(6)} + 1 = m + 1$ equations over G_κ^t .
- (ii) The groups G_κ are cyclic: $G_\kappa = \frac{n}{\gcd(d_\kappa, n)} \cdot \mathbb{Z}_n \subseteq \mathbb{Z}_n$. J_κ can be seen as a homomorphism system (J_κ, G_κ^t) .
- (iii) J_κ can be displayed as $\begin{pmatrix} I_{t(m+1)} & B^{(3)} \end{pmatrix}$ for certain $B^{(3)}$ depending on κ and with dimensions $t(m+1) \times t((m^{(5)} + 2) - (m+1))$. All the rows $B_{[i]}^{(3)}$ from $B^{(3)}$ have $\gcd(B_{[i]}^{(3)}, |G_\kappa|) = 1$. Moreover, the block of t consecutive rows $\mathcal{B}_i^{(3)} = B_{[(i-1)t+1, \dots, (i-1)t+t]}^{(3)}$, $i \in [1, m+1]$, is such that $\gcd(D_t(\mathcal{B}_i^{(3)}), |G_\kappa|) = 1$. Even more, $\gcd(D_t(\mathcal{B}_i^{(3)}), n) = 1$

Remark 27. For any $\kappa \in \Upsilon$, the homomorphism

$$f_\kappa : S(J_\kappa, G_\kappa^t) \longrightarrow S(A^{(6)}, G_\kappa^t) \subset S(A^{(6)}, \mathbb{Z}_n^t)$$

$$\begin{pmatrix} (x_{(1,1)}, \dots, x_{(1,t)}) \\ \vdots \\ (x_{(m^{(j)},1)}, \dots, x_{(m^{(j)},t)}) \end{pmatrix} \longmapsto \begin{pmatrix} (d_{1,1} x_{(1,1)}, \dots, d_{1,t} x_{(1,t)}) \\ \vdots \\ (d_{k^{(6)},1} x_{(k^{(6)},1)}, \dots, d_{k^{(6)},t} x_{(k^{(6)},t)}) \\ (x_{(k^{(j)}+1,1)}, \dots, x_{(k^{(j)}+1,t)}) \\ \vdots \\ (x_{(m^{(j)}-1,1)}, \dots, x_{(m^{(j)}-1,t)}) \end{pmatrix}$$

is surjective and $|G_\kappa^t|$ -to-1.

¹⁶Observe that, if $\gcd(d_{i,j}, n) = 1$, then $G_{(i,j)} = \{0\}$.

Proof of Remark 27. The variables with indices $[k^{(6)}+1, m^{(6)}]$ from $(A^{(6)}, G_\kappa^t)$ parameterize the solution set $(A^{(6)}, G_\kappa^t)$. This is, any choice of $x_{(k^{(6)}+1, \cdot)}, \dots, x_{(m^{(6)}, \cdot)} \in G_\kappa^t$ provide a unique solution to $(A^{(6)}, G_\kappa^t)$. The same holds true for the variables indexed by $[k^{(j)}+1, m^{(j)}]$ in the system (J_κ, G_κ^t) .

Assume $(i, j) \in \Upsilon \setminus \{(1, 0)\}$. The (i, j) -th equation for $(A^{(6)}, G_\kappa^t)$ can be written as

$$x_{(i,j)} + d_{i,j} B_{[(i-1)t+j]}^{(2)} \cdot (x_{(k^{(6)}+1,j)}, \dots, x_{(m^{(6)},j)})^\top = 0$$

On the other hand, the (i, j) -th equation in (J_κ, G_κ^t) is

$$\left\{ \begin{array}{l} y_{(i,j)} + B_{\kappa,[(i-1)t+j]}^{(3)} \cdot (y_{(k^{(j)}+1,j)}, \dots, y_{(m^{(j)},j)})^\top = \\ = y_{(i,j)} + B_{[(i-1)t+j]}^{(2)} \cdot (y_{(k^{(j)}+1,j)}, \dots, y_{(m^{(j)}-1,j)})^\top = 0 \end{array} \right\} \quad \text{if } (i, j) \neq \kappa$$

$$\left\{ \begin{array}{l} y_{(i,j)} + B_{\kappa,[(i-1)t+j]}^{(3)} \cdot (y_{(k^{(j)}+1,j)}, \dots, y_{(m^{(j)},j)})^\top = \\ y_{(i,j)} + d_{i,j} B_{[(i-1)t+j]}^{(2)} \cdot (y_{(k^{(j)}+1,j)}, \dots, y_{(m^{(j)}-1,j)})^\top - y_{(m^{(j)},j)} = 0 \end{array} \right\} \quad \text{if } (i, j) = \kappa.$$

Therefore, if we let $(x_{(k^{(6)}+1, \cdot)}, \dots, x_{(m^{(6)}, \cdot)}) = (y_{(k^{(j)}+1, \cdot)}, \dots, y_{(m^{(j)}-1, \cdot)})$, the variables $y_{(i,j)}$ and $x_{(i,j)}$ are such that $d_{i,j} y_{(i,j)} = x_{(i,j)}$ for $(i, j) \neq \kappa$. If $(i, j) = \kappa$, then $d_{i,j} G_\kappa = 0$ and $(\mathbf{x})_\kappa = 0 = d_{i,j} (\mathbf{y})_\kappa$ for any pair of solutions $\mathbf{x} \in S(A^{(6)}, G_\kappa^t)$ and $\mathbf{y} \in S(J_\kappa, G_\kappa^t)$. This shows that the map f_κ exists.

Since f_κ maps the subset of parameterizing variables $(y_{(k^{(j)}+1, \cdot)}, \dots, y_{(m^{(j)}-1, \cdot)})$ to the parameterizing variables $(x_{(k^{(6)}+1, \cdot)}, \dots, x_{(m^{(6)}, \cdot)})$ using the identity map, f_κ is surjective. Moreover, since the image by f_κ is independent of the variable $y_{(m^{(j)}, \cdot)}$, the map is $|G_\kappa^t|$ -to-1. \square

In the following part, Section 5.6, we adapt to the case of homomorphisms matrices the properties of the n -circular matrices used to show 1-strong-representations in [21] and [3, 19].¹⁷ In particular, Proposition 28 in Section 5.6 constructs, given an n -circular matrix, a matrix C with good properties for the representation.

In Section 5.7 an n -circular matrix \bar{J}_κ is constructed for each matrix J_κ , $\kappa \in \Upsilon$. The final construction of the 1-strong-representation is conducted in Section 5.8; it involves combining the matrices \bar{J}_κ , $\kappa \in \Upsilon$, in a single matrix $A^{(\tau)}$, as well as combining all the matrices C_κ , provided by Section 5.6, in a single matrix C .

5.6 n -circular matrices and properties

An integer matrix A formed by $k \times m$ square blocks, $m \geq k$, is said to be *block n -circular* if all the matrices formed by k consecutive columns of blocks of A , (A^i, \dots, A^{i+k-1}) (considering the indices modulo m) have determinant coprime with n . A matrix is called standard n -circular if it is n -circular and with the shape $(I_k \ B)$. When the size of the blocks is one this definition coincides with the one provided in [21, Definition 3]. The properties of the n -circular matrices described in Proposition 28 are used in the construction of the representation described in Section 5.8.

Proposition 28. *Let A be a $kt \times mt$ integer matrix, $m \geq k$, formed by km blocks of size $t \times t$. Assume that A is block n -circular. Then there exists a $m \times m$ block integer matrix $C = (C_{i,j})$, each block of size $t \times t$ and $(i, j) \in [1, m]^2$, with the following properties.*

i. $AC = 0$

ii. *The i -th row of $t \times t$ blocks is such that $C_{i,j} = 0$, for $j \in \{i+k+1, \dots, i-1\}$ with indices modulo m . So, the matrix looks like*

$$C = \begin{pmatrix} * & * & * & * & 0 & 0 \\ 0 & * & * & * & * & 0 \\ 0 & 0 & * & * & * & * \\ * & 0 & 0 & * & * & * \\ * & * & 0 & 0 & * & * \\ * & * & * & 0 & 0 & * \end{pmatrix}$$

iii. $\gcd(\det(C_{i,i}), n) = 1$ and $\gcd(\det(C_{i,k+i}), n) = 1$ for all $i \in [1, m]$ with indices taken modulo m .

¹⁷Although the final representation to prove [21, Theorem 1] is not strong, the one established in [21, Lemma 4] has the strong property.

Proof of Proposition 28. Consider the square matrix formed by the column blocks $A^{[i,i+k-1]} = (A^{[i]}, \dots, A^{[i+k-1]})$ with $i \in [1, m]$. By assumption $A^{[i,i+k-1]}$ is a square non-singular matrix as it has non-zero determiniant. For the j -th vector in the column block $A^{[i+k]}, A^{[i+k],j}$, with $j \in [1, t]$, we can find rational coefficients $b_{(i-1)t+w, (i+k-1)t+j}$, $w \in [1, kt]$, with

$$A^{[i+k],j} = \sum_{w \in [1, kt]} b_{(i-1)t+w, (i+k-1)t+j} A^{[i,i+k-1],w}$$

where $A^{[i,i+k-1],w}$ stands for the w -th column in $A^{[i,i+k-1]}$ and corresponds to the $((i-1)t+w)$ -th column in A . Moreover, since the determinant is coprime with n , there exists an integer $c_{(i+k-1)t+j, (i+k-1)t+j}$, coprime with n , such that

$$-c_{(i+k-1)t+j, (i+k-1)t+j} A^{[i+k],j} = \sum_{w \in [1, kt]} c_{(i-1)t+w, (i+k-1)t+j} A^{[i,i+k-1],w} \quad (6)$$

where, for $w \in [1, kt]$, $c_{(i-1)t+w, (i+k-1)t+j} = -c_{(i+k-1)t+j, (i+k-1)t+j} b_{(i-1)t+w, (i+k-1)t+j}$ are integers.

The coefficients of the matrix C are

- (a) c_{w_1, w_2} whenever the subscripts (w_1, w_2) coincide with one the c 's found in the relations given by (6) for the mk column vectors of A .
- (b) 0 otherwise.

Consider the matrix C as divided into $t \times t$ blocks $\mathcal{C}_{\cdot, \cdot}$. C satisfy property ii. Indeed, given a column of C indexed by $j = j_1 t + j_2$, with $j_1 \in [0, m-1]$ and $j_2 \in [1, t]$, the indices i of the rows involved in the relations given by (6) satisfy $i \in [(j_1 - k)t, (j_1 + 1)t]$.

The relation (6) can be rearranged as

$$0 = c_{(i+k-1)t+j, (i+k-1)t+j} A^{[i+k],j} + \sum_{w \in [1, kt]} c_{(i-1)t+w, (i+k-1)t+j} A^{[i,i+k-1],w}$$

and can be extended to $\sum_{w \in [1, tm]} c_{w, (i+k-1)t+j} A^l$ considering that all the other c 's that appear in the sum are zero by (b). Thus i is satisfied.

Observe that $\mathcal{C}_{i,i}$ is a diagonal matrix where all the elements in the diagonal are coprime with n . Hence the first part of property iii is satisfied. To show the second part observe that, for each $i \in [1, m]$, indices modulo m ,

$$(A^{[i-k]} \quad \dots \quad A^{[i-1]}) \begin{pmatrix} 0 & 0 & \dots & 0 & \mathcal{C}_{i-k,i} \\ & & & I_{t(k-1)} & \vdots \\ & & & & \mathcal{C}_{i-1,i} \end{pmatrix} = (A^{[i-k+1]} \quad \dots \quad A^{[i-1]} \quad \overline{A}^{[i]})$$

where the columns of $\overline{A}^{[i]}$ are multiples of the columns of $A^{[i]}$ by (6). Indeed, $\overline{A}^{[i],j} = -c_{(i-1)t+j, (i-1)t+j} A^{[i],j}$. Therefore

$$\det(A^{[i-k+1]} \quad \dots \quad A^{[i-1]} \quad \overline{A}^{[i]}) = \det(A^{[i-k+1]} \quad \dots \quad A^{[i-1]} \quad A^{[i]}) \prod_{j \in [1, t]} c_{(i-1)t+j, (i-1)t+j}$$

which is a product of integers coprime with n . Since

$$\det \begin{pmatrix} 0 & 0 & \dots & 0 & \mathcal{C}_{i-k,i} \\ & & & I_{t(k-1)} & \vdots \\ & & & & \mathcal{C}_{i-1,i} \end{pmatrix} = \pm \det(\mathcal{C}_{i-k,i})$$

and $(\pm \det(\mathcal{C}_{i-k,i})) \cdot \det(A^{[i-k]} \quad \dots \quad A^{[i-1]}) = \det(A^{[i-k+1]} \quad \dots \quad A^{[i-1]} \quad \overline{A}^{[i]})$, then $\det(\mathcal{C}_{i-k,i})$ is an integer coprime with n . This proves the second part of iii and finalizes the proof of the proposition. \square

5.7 Construction of the n -circular matrix

Let n be a positive integer and let G be an abelian group of order n . For our purposes, we can assume $G = \mathbb{Z}_n$. Let A be a $kt \times mt$ matrix $A = (I_{tk} \quad B)$ though of as built with km blocks of size $t \times t$. Moreover, we shall assume that $\gcd(D_t(\mathcal{B}_i), n) = 1$, where \mathcal{B}_i is the i -th block of t rows of the submatrix B , $i \in [1, k]$. In this section we build a $tk^{(9)} \times tm^{(9)}$ integer matrix $A^{(9)} = (I_{tk^{(9)}} \quad B)$ such that:

- $(A^{(9)}, G^t)$ is 1-auto-equivalent to (A, G^t) .
- $A^{(9)}$ is n -circular with blocks of length 1, hence n -circular with blocks of size t .

We enlarge the $t \times (m-k)t$ matrix \mathcal{B}_i using Lemma 23 to the $(m-k)t \times (m-k)t$ matrix

$$\mathcal{B}_i^{(4)} = \begin{pmatrix} \mathcal{B}_i \\ M_i \end{pmatrix} \text{ with } \det \begin{pmatrix} \mathcal{B}_i \\ M_i \end{pmatrix} = 1.$$

By adding some new variables taking values in G , $\mathcal{A}_i = \begin{pmatrix} I_t & \mathcal{B}_i \end{pmatrix}$ turns into the matrix denoted by $\mathcal{A}_i^{(8)}$ with

$$\mathcal{A}_i^{(8)} = \begin{pmatrix} I_t & 0 & \mathcal{B}_i \\ 0 & I_{(m-k-1)t} & M_i \end{pmatrix} = \begin{pmatrix} I_{(m-k)t} & \mathcal{B}_i^{(4)} \end{pmatrix}$$

Let us denote by $B^{(4)}$ the matrix formed by attaching together all the rows in $\{\mathcal{B}_i^{(4)}\}_{i \in [1, k]}$

$$B^{(4)} = \begin{pmatrix} \mathcal{B}_1^{(4)} \\ \vdots \\ \mathcal{B}_k^{(4)} \end{pmatrix}.$$

Denote by $A^{(8)}$ the matrix $A^{(8)} = \begin{pmatrix} I_{k(m-k)t} & B^{(4)} \end{pmatrix}$. The variables added with respect to A take values over the whole G . The system $(A^{(8)}, G)$ and $(A^{(8)}, G^t)$ are 1-auto-equivalent to (A, G) and (A, G^t) respectively.

A Lemma for the building blocks. Lemma 29 improves [21, Lemma 11] so that each block can be constructed by adding a linear number of rows with respect to the original number of columns.

Lemma 29. *Let n and r be positive integers and let M be an $r \times r$ integer matrix with determinant coprime with n . There are $r \times r$ integer matrices S and T such that*

$$M' = \begin{pmatrix} I_r \\ S \\ M \\ T \\ I_r \end{pmatrix}$$

is a $5r \times r$ integer matrix with the property that each $r \times r$ submatrix of M' , consisting of r consecutive rows, has a determinant coprime with n .

Proof of Lemma 29. We detail the construction of T . Let us define the matrices $r-i \times r$ matrices

$$M^i = \begin{pmatrix} M_{i+1}^i \\ \vdots \\ M_r^i \end{pmatrix}, \quad i \in [0, r-1],$$

together with the rows T_{i+1} inductively. Let $M^0 = M$. Let $d_i = \gcd(M_{i,i}^{i-1}, \dots, M_{r,i}^{i-1})$, $i \in [1, r]$, be the greatest common divisor of the column $M_{\cdot, i}^{i-1}$. Let

$$T_i = \lambda_i^i M_i^{i-1} + \dots + \lambda_r^i M_r^{i-1}$$

where $\lambda_i^i, \dots, \lambda_r^i$ are such that

$$\lambda_i^i M_{i,i}^{i-1} + \dots + \lambda_r^i M_{r,i}^{i-1} = d_i \tag{7}$$

and where λ_i^i is some prime, p_i , larger than n . This p_i exists, subjected to the constrain (7), by the Dirichlet theorem regarding the containment of infinitely many primes in the arithmetic progressions $a+b\mathbb{Z}$ with $\gcd(a, b) = 1$. Observe that

$$\det \begin{pmatrix} M_i^{i-1} \\ \vdots \\ M_r^{i-1} \\ T_1 \\ \vdots \\ T_{i-1} \end{pmatrix} = p_i \det \begin{pmatrix} M_{i+1}^{i-1} \\ \vdots \\ M_r^{i-1} \\ T_1 \\ \vdots \\ T_i \end{pmatrix}$$

The rows of the matrix M^i , denoted by M_j^i , are $M_j^i = M_j^{i-1} - (M_{j,i}^{i-1}/d_i)T_i$, for $j \in [i+1, r]$ and with $T_0 = \mathbf{0}$. The first i columns of M^i are the zero columns.

Observe that $\gcd(d_i, n) = 1$ as

$$\det \begin{pmatrix} M_{i+1}^{i-1} \\ \vdots \\ M_r^{i-1} \\ T_1 \\ \vdots \\ T_i \end{pmatrix} = \det \begin{pmatrix} M_{i+1}^i \\ \vdots \\ M_r^i \\ T_1 \\ \vdots \\ T_i \end{pmatrix}$$

is coprime with n and the original matrix M has determinant coprime with n . Therefore the equivalent matrix

$$\begin{pmatrix} M_{i+1}^i \\ \vdots \\ M_r^i \\ T_1 \\ \vdots \\ T_i \end{pmatrix} \sim \begin{pmatrix} M_{i+1} \\ \vdots \\ M_r \\ T_1 \\ \vdots \\ T_i \end{pmatrix}$$

also has a determinant coprime with n . This shows the property regarding the coprimality of the determinant of consecutive rows for the first r rows constructed in this way, T_1, \dots, T_r . Observe that

$$T_i = (\overbrace{0 \cdots 0}^{i-1} d_i * \cdots *).$$

Since each d_i is coprime with n , we can add the identity matrix after the matrix T and the claimed properties are satisfied. The matrix S is built similarly but we start from the last column and we construct a lower diagonal matrix S . \square

Attaching building blocks. We use Lemma 29 on each matrix $\mathcal{B}_i^{(4)}$ to obtain matrices

$$\begin{pmatrix} I_{t(m-k)} \\ S_i \\ \mathcal{B}_i^{(4)} \\ T_i \\ I_{t(m-k)} \end{pmatrix} = \begin{pmatrix} I_{t(m-k)} \\ \mathcal{B}_i^{(5)} \\ I_{t(m-k)} \end{pmatrix}$$

that are put together into a large matrix

$$A^{(9)} = \begin{pmatrix} I_{t(m-k)} \\ \mathcal{B}_1^{(5)} \\ I_{t(m-k)} \\ \vdots \\ I_{t(m-k)} \\ \mathcal{B}_k^{(5)} \\ I_{t(m-k)} \end{pmatrix} = (I_{(4k+1)t(m-k)} \quad B^{(9)})$$

that is n -circular. This is, any $r = (4k+1)t(m-k)$ consecutive columns form a matrix with determinant coprime with n . Indeed, the matrix formed by the first r columns is the identity matrix. On the other cases, some columns of the left most identity matrix I_r are selected, along with some other columns from the $B^{(9)}$ part. Therefore the determinant is, up to a sign, the determinant of the submatrix formed by the columns selected in $B^{(9)}$ and the rows corresponding to the indices of the columns not picked from I_r . If the set of columns selected are consecutive and contains all the columns of $B^{(9)}$, the determinant is coprime with n as so is the determinant formed with $t(m-k)$ consecutive rows from $B^{(9)}$. Since the first and the last square blocks of $B^{(9)}$ are identity matrices, the remaining cases are shown.

For $G^t = \mathbb{Z}_n^t$, the equations induced by the new rows in $B^{(9)}$ with respect to $B^{(4)}$ are

$$x_i + \sum_{j=1}^{t(m-k)} B_{i,j}^{(9)} x_{r+j} = 0, \quad x_i \in \mathbb{Z}_n.$$

Therefore, $(A^{(9)}, G^t)$ is a $k^{(9)} \times m^{(9)}$ homomorphism system 1-auto-equivalent to $(A^{(8)}, G^t)$.

Remark 30. The system $(A^{(9)}, \mathbb{Z}_n^t)$ is 1-auto-equivalent to (A, \mathbb{Z}_n^t) by projecting onto the original coordinates using maps ϕ_i equal to the identity map. Indeed, any solution to (A, \mathbb{Z}_n^t) can be extended uniquely to a solution in $(A^{(9)}, \mathbb{Z}_n^t)$ as the last $m - k$ variables in both systems parameterize the solutions in both cases.

5.8 Final composition for $\gamma \neq 1$ and representation for $(A^{(7)}, G)$

Joining the matrices and groups. Let $\{\bar{J}_\kappa\}_{\kappa \in \Upsilon}$, be the n -circular integer matrices obtained from $\{J_\kappa = (I_{t(m+1)} \quad B_\kappa^{(3)})\}_{\kappa \in \Upsilon}$ using the procedure in Section 5.7. This applies by (iii) in Section 5.5 regarding $\gcd(D_t(\mathcal{B}_{\kappa,i}^{(3)}), n) = 1$. All the matrices \bar{J}_κ have the same dimensions $tk^{(j)} = (4k^{(j)} + 1)t(m^{(j)} - k^{(j)})$, $tm^{(j)} = (4k^{(j)} + 2)t(m^{(j)} - k^{(j)})$ over G_κ .

Consider Υ to be ordered lexicographically; given $(\kappa_1, \kappa_2), (\kappa_3, \kappa_4) \in \Upsilon$, $(\kappa_1, \kappa_2) < (\kappa_3, \kappa_4)$ if and only if $\kappa_1 m + \kappa_2 < \kappa_3 m + \kappa_4$. The columns of the matrix $A^{(7)}$ correspond to the columns $(\bar{J}_\kappa)_v$ using the lexicographical order for the ordered set $[1, tm^{(j)}] \times \Upsilon \ni (v, \kappa)$. The rows $A^{(7)}$ correspond to the rows $(\bar{J}_\kappa)_w$ using the lexicographically ordered set $[1, tk^{(j)}] \times \Upsilon \ni (w, \kappa)$. The coefficients of $A^{(7)}$ are zero wherever the intersection of a column and a row does not appear in any of the matrices \bar{J}_κ . This is, the (i, j) element of $A^{(7)}$ is given by

$$(A^{(7)})_{i,j} = \begin{cases} (\bar{J}_\kappa)_{\lambda_1, \mu_1} & \text{if } \begin{cases} i = (\lambda_1 - 1)(1 + mt) + 1 + (\kappa_1 - 1)t + \kappa_2 \\ j = (\mu_1 - 1)(1 + mt) + 1 + (\kappa_1 - 1)t + \kappa_2 \end{cases} \\ & \text{for some } \begin{cases} \lambda_1 \in [1, tk^{(j)}], \mu_1 \in [1, tm^{(j)}] \\ \kappa = (\kappa_1, \kappa_2) \in \Upsilon \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\overline{A^{(7)}}$ to be the block-diagonal matrix containing the matrices $\{\bar{J}_\kappa\}_{\kappa \in \Upsilon}$ as the blocks in the diagonal. The matrix $A^{(7)}$ can be seen as an appropriate permutation of rows and columns of the block matrix $\overline{A^{(7)}}$. Let P_1 and P_2 denote, respectively, the row and column permutations so that $A^{(7)} = P_1 \overline{A^{(7)}} P_2$.

$A^{(7)}$ can be considered as formed by $t^2 k^{(j)} m^{(j)}$ blocks of size $(1 + tm) \times (1 + tm)$ over the groups $G = \prod_{\kappa \in \Upsilon} G_\kappa$. Furthermore, t^2 of the $(1 + tm) \times (1 + tm)$ blocks can be grouped in a single block of size $t(1 + tm) \times t(1 + tm)$. This allows us to interpret $A^{(7)}$ as formed by $k^{(9)} m^{(9)}$ blocks of size $t(1 + tm) \times t(1 + tm)$ over the groups $G^t = (\prod_{\kappa \in \Upsilon} G_\kappa)^t$. Therefore, if we denote $k^{(7)} = k^{(j)}$ and $m^{(7)} = m^{(j)}$, $A^{(7)}$ can be considered as a $k^{(7)} \times m^{(7)}$ homomorphism system over G^t denoted by $(A^{(7)}, G^t)$. $(A^{(7)}, G^t)$ has the particularity that the solution set of the system $A^{(7)}, S(A^{(7)}, G^t)$, is the cartesian product of the solution sets $\{S(\bar{J}_\kappa, G_\kappa^t)\}_{\kappa \in \Upsilon}$.

Matrix C for $A^{(7)}$. Since each of the matrices $\{\bar{J}_\kappa\}_{\kappa \in \Upsilon}$ is n -circular with block size 1, we use Proposition 28 to find band-shape matrices C_κ related to \bar{J}_κ for $\kappa \in \Upsilon$.

All the $\{C_\kappa\}_{\kappa \in \Upsilon}$ are joined into a single C fulfilling the properties stated in Proposition 28 for $A = A^{(7)}$. Indeed, let \overline{C} be the block matrix with the matrices $\{C_\kappa\}_{\kappa \in \Upsilon}$ in the diagonal and zeros everywhere else. Observe that $\overline{A^{(7)}} \overline{C} = \mathbf{0}$. Let $C = P_2^{-1} \overline{C} P_2$, where P_2 is the column permutation from $\overline{A^{(7)}}$ into $A^{(7)}$. Then the equality $A^{(7)} C = \mathbf{0}$ follows.

If we group the consecutive rows and columns of C by blocks of size $t(1 + mt) \times t(1 + mt)$, then C can be considered as a $(t(1 + mt))^2$ -sized-block matrix $C = (C_{i,j})$ with $(i, j) \in [1, m^{(7)}] \times [1, m^{(7)}]$. Moreover, C has the band-shape inherited from $\{C_\kappa\}_{\kappa \in \Upsilon}$. In particular,

- $C_{i,i}$, $i \in [1, m^{(7)}]$, is a $t(1 + mt) \times t(1 + mt)$ upper triangular matrix where each coefficient in the diagonal is coprime with n .
- $C_{i,i+k^{(7)}}$, $i \in [1, m^{(7)}]$, is a $t(1 + mt) \times t(1 + mt)$ lower triangular matrix where each coefficient in the diagonal is coprime with the order of the group on which it is acting.¹⁸

¹⁸The matrix is lower triangular and not only block lower triangular (with the blocks in the diagonal having determinant coprime with n). Indeed, the matrices \bar{J}_κ built using Lemma 29 are n -circular for blocks of size 1.

- $\mathcal{C}_{i,j} = 0$ for $i \in [1, m^{(\tau)}]$ and $j \in [i + k^{(\tau)} + 1, \dots, i - 1]$, indices modulo $m^{(\tau)}$.

Strongly 1-representation for $(A^{(\tau)}, G^t)$. We use a similar machinery as in [21, Lemma 4] to construct, assuming $m^{(\tau)} \geq k^{(\tau)} + 2$, a strongly 1-representation for $(A^{(\tau)}, G)$ denoted by (K, H) .

Let the hypergraph K have the vertex set $V(K) = \left(\prod_{\kappa \in \Upsilon} G_\kappa\right)^t \times [1, m^{(\tau)}]$. H has $[1, m^{(\tau)}]$ as its vertex set and $m^{(\tau)}$ edges $e_i = \{i, \dots, i + k^{(\tau)}\} \bmod m^{(\tau)}$ for $i \in [1, m^{(\tau)}]$ with e_i coloured i . Since $m^{(\tau)} \geq k^{(\tau)} + 2$, H has $m^{(\tau)}$ pair-wise distinct edges.

The edges in $K^{(\tau)}$ form a $m^{(\tau)}$ -partite $(k^{(\tau)} + 1)$ -uniform hypergraph. The edge $e_i = \{g_i, \dots, g_{i+k^{(\tau)}}\}$, with $g_j \in G^t \times \{j\} \subset V(K)$, is coloured i and is labelled g if and only

$$\sum_{j=i}^{i+k^{(\tau)}} \mathcal{C}_{i,j}(g_j) = g.$$

These labels on the edges define a labelling function $l : E(K) \rightarrow G^t$. Furthermore, the uniformity of the edges is $k^{(\tau)} + 1$ which is bounded by $m^{(\tau)} = |V(H)|$. This shows RP1 from Definition 10 with $\chi_1 = m^{(\tau)}$.

For RP2, observe that all the copies of H in K should contain one, and exactly one, vertex in each of the vertex clusters $G^t \times \{j\}$ in $V(K)$. Given $H_0 \in C(H, K)$ with $V(H_0) = \{g_1, \dots, g_{m^{(\tau)}}\}$, the labels on the edges $e_1, \dots, e_{m^{(\tau)}}$ of H_0 are given by $(l(e_1), \dots, l(e_{m^{(\tau)}}))^T = C(g_1, \dots, g_{m^{(\tau)}})^T$. Since $A^{(\tau)}C = 0$, then $(l(e_1), \dots, l(e_{m^{(\tau)}})) \in S(A^{(\tau)}, G^t)$. Let $Q = \{1\}$, $r_q(H_0) = 1$ for each $H_0 \in C(H, K)$ and $p = 1$. Then we define

$$r : C(H, K) \longrightarrow S(A^{(\tau)}, G^t) \times Q$$

$$H_0 = \{e_1, \dots, e_{m^{(\tau)}}\} \longmapsto (r_0(H_0), r_q(H_0)) = ((l(e_1), \dots, l(e_{m^{(\tau)}})), 1).$$

We claim that r induces a strong 1-representation for some c bounded by a function of $m^{(\tau)}$. Let $(\mathbf{x}_1, \dots, \mathbf{x}_{m^{(\tau)}}) = \mathbf{x} \in S(A^{(\tau)}, G^t)$ be a solution. For each $i \in [1, m^{(\tau)}]$, there are $|G^t|^{k^{(\tau)}}$ edges labelled \mathbf{x}_i . Indeed, as $\mathcal{C}_{i,i}$ is a block with determinant coprime with the order of the group G^t , for any choice of the vertices $\{g_{i+1}, \dots, g_{i+k^{(\tau)}}\}$ there is a unique vertex $g_i \in G \times \{i\}$ such that $\sum_{j=i}^{i+k^{(\tau)}} \mathcal{C}_{i,j}(g_j) = \mathbf{x}_i$, namely $g_i = \mathcal{C}_{i,i}^{-1}(\mathbf{x}_i - \sum_{j=i+1}^{i+k^{(\tau)}} \mathcal{C}_{i,j}(g_j))$.¹⁹

Given the solution \mathbf{x} and an edge e_i coloured i and labelled \mathbf{x}_i in K , there is a unique H_0 , copy of H in K , with $r_0(H_0) = \mathbf{x}$ and $e_i \in H_0$. Indeed, if e_i has the vertices $\{g_i, \dots, g_{i+k^{(\tau)}}\}$ as its support, then there exists a unique $g_{i-1} \in G^t \times \{i-1\} \subset V(K)$ such that $l(\{g_{i-1}, \dots, g_{i+k^{(\tau)-1}}\}) = \mathbf{x}_{i-1}$. This process can be repeated a total of $m^{(\tau)} - k^{(\tau)} - 1$ times. Indeed, the vertices $g_{i-1}, g_{i-2}, \dots, g_{i+k^{(\tau)}+1}$ that complete, together with $g_i, \dots, g_{i+k^{(\tau)}}$, a copy of H in K can be uniquely determined using the coordinate values $\mathbf{x}_{i-1}, \dots, \mathbf{x}_{i+k^{(\tau)}+1}$ of the solution \mathbf{x} and a subset of the previously determined vertices.

Let H_0 denoted this copy of H in K . By the existence of r , $r_0(H_0) \in S(A^{(\tau)}, G^t)$. Even more, we claim that $r_0(H_0) = \mathbf{x}$. Indeed, by the n -circularity of $A^{(\tau)}$, any $m^{(\tau)} - k^{(\tau)}$ consecutive values of $\{\mathbf{x}_j\}$ determines the solution. The copy H_0 has been constructed so that it contains edges labelled with the $m^{(\tau)} - k^{(\tau)}$ consecutive values $\mathbf{x}_i, \mathbf{x}_{i-1}, \dots, \mathbf{x}_{i+k^{(\tau)}+1}$ and coloured appropriately with $\{i, i-1, \dots, i+k^{(\tau)}+1\}$. Since the only solution $\mathbf{y} \in S(A^{(\tau)}, G^t)$ that satisfies $(\mathbf{y})_j = \mathbf{x}_j$ for $j \in \{i, i-1, \dots, i+k^{(\tau)}+1\}$ is $\mathbf{y} = \mathbf{x}$ and $r_0(H_0) \in S(A^{(\tau)}, G^t)$, then $r_0(H_0) = \mathbf{x}$.

Hence, given $\mathbf{x} \in S(A^{(\tau)}, G^t)$ and $i \in [1, m^{(\tau)}]$, each copy of e_i coloured i and labelled \mathbf{x}_i can be extended to a unique copy of H in K related to \mathbf{x} . This shows RP3 and RP4. Moreover, the number of copies of H related to \mathbf{x} in K is

$$|r^{-1}(\mathbf{x}, 1)| = |G^t|^{k^{(\tau)}} = \frac{|G^t|^{k^{(\tau)}+1}}{|G^t|} = \left(\frac{1}{m^{(\tau)}}\right)^{k^{(\tau)}+1} \frac{|K|^{k^{(\tau)}+1}}{|G^t|} = c \frac{|K|^{k^{(\tau)}+1}}{|G^t|}$$

as there are $|G^t|^{k^{(\tau)}}$ edges labelled $(\mathbf{x})_i$ for any $i \in [1, m^{(\tau)}]$. This shows RP2 and finishes the strong 1-representation for $(A^{(\tau)}, G^t)$ by (K, H) .

Relation with previous systems. Observe that each of the systems $\{(J_\kappa, G_\kappa^t)\}_{\kappa \in \Upsilon}$ have one more equation and two more variables than $(A^{(6)}, \mathbb{Z}_n^t)$ and that $(A^{(\tau)}, G^t)$ have the same number of equations and variable as any $(\bar{J}_\kappa, G_\kappa^t)$. Thus, $A^{(\tau)}$ has dimensions $k^{(\tau)} = (4k^{(6)} + 5)(m^{(6)} - k^{(6)} + 1)$ and $m^{(\tau)} = (4k^{(6)} + 6)(m^{(6)} - k^{(6)} + 1)$ over G^t .

¹⁹All the rational numbers appearing in $\mathcal{C}_{i,i}^{-1}$ have denominators co-prime with $|G^t|$, hence inducing automorphisms in G .

If x_i is a variable in $(A^{(7)}, G^t)$, then it can be decomposed into t variables $x_{(i,j)} \in G$ with $j \in [1, t]$. Furthermore, each $x_{(i,j)}$ can be understood as formed by $mt + 1$ variables $x_{(i,j),\kappa} \in G_\kappa$, where $\kappa \in \Upsilon$.

Remark 31. The system $(A^{(7)}, G^t)$ is μ -equivalent to $(A^{(6)}, \mathbb{Z}_n^t)$ with the injection

$$\sigma : [1, m^{(6)}] \longrightarrow [1, m^{(7)}]$$

$$i \longmapsto \begin{cases} 4(i-1)(m^{(6)} - k^{(6)} + 1) + 3(m^{(6)} - k^{(6)} + 1) + 1 & \text{if } i \in [1, k^{(6)}] \\ (4k^{(6)} + 5)(m^{(6)} - k^{(6)} + 1) + i - k^{(6)} & \text{if } i \in [k^{(6)} + 1, m^{(6)}] \end{cases}$$

and

$$\phi : S(A^{(7)}, G^t) \longrightarrow S(A^{(6)}, \mathbb{Z}_n^t)$$

$$(x_1, \dots, x_{m^{(6)}}) \longmapsto (\phi_1(x_{\sigma(1)}), \dots, \phi_{m^{(6)}}(x_{\sigma(m^{(6)})}))$$

with

$$\phi_i(x_{(\sigma(i),j)}) = \begin{cases} d_{i,j} \sum_{\kappa \in \Upsilon} x_{(\sigma(i),j),\kappa} & \text{for } i \in [1, k^{(6)}] \\ \sum_{\kappa \in \Upsilon} x_{(\sigma(i),j),\kappa} & \text{for } i \in [k^{(6)} + 1, m^{(6)}] \end{cases},$$

where $d_{i,j}$ is the greatest common divisor of the j -th row of the block $\mathcal{B}_i^{(1)}$ from $A^{(6)} = (I_{tm} \quad B^{(1)})$.²⁰

Additionally, for any $\mathbf{x} \in S(A^{(6)}, \mathbb{Z}_n^t)$ and $i \in [1, m^{(6)}]$, the number of $\mathbf{y} \in S(A^{(7)}, G^t)$ with $\phi(\mathbf{y}) = \mathbf{x}$ and fixed $(\mathbf{y})_{\sigma(i)}$ is independent of the $(\mathbf{y})_{\sigma(i)} \in \phi_i^{-1}((\mathbf{x})_i)$.

Proof of Remark 31. The system $(A^{(7)}, G^t)$ is formed by joining the systems $\{(\bar{J}_\kappa, G_\kappa^t)\}_{\kappa \in \Upsilon}$ together. The $k^{(j')} \times m^{(j')}$ system $(\bar{J}_\kappa, G_\kappa^t)$ is 1-auto-equivalent to (J_κ, G_κ^t) by Remark 30 for any $\kappa \in \Upsilon$.²¹ Let ϕ'_κ be the map defining the 1-auto-equivalence from $S(\bar{J}_\kappa, G_\kappa^t)$ to $S(J_\kappa, G_\kappa^t)$.

Any solution $\mathbf{x} \in S(A^{(7)}, G^t)$ induces $(mt + 1)$ solutions $\bar{\mathbf{x}}_\kappa \in S(\bar{J}_\kappa, G_\kappa^t)$ and vice-versa. By the 1-auto-equivalence, these solutions can be seen in (J_κ, G_κ^t) considering $\mathbf{x}_\kappa = \phi'_\kappa(\bar{\mathbf{x}}_\kappa) \in S(J_\kappa, G_\kappa^t)$. We use the maps f_κ from Remark 27 to conclude that

$$\phi(\mathbf{x}) = \sum_{\kappa \in \Upsilon} f_\kappa(\mathbf{x}_\kappa) \in S(A^{(6)}, \mathbb{Z}_n^t)$$

as ϕ is the sum over Υ of the compositions of the homomorphisms f_κ with the ϕ'_κ .

Since the homomorphism $f_{(1,0)}$ associated to $(J_{(1,0)}, \mathbb{Z}_n^t) = (J_{(1,0)}, G_{(1,0)}^t)$ is surjective, so is ϕ .²² As ϕ is a homomorphism, ϕ is μ -to-1 for $\mu = |S(A^{(7)}, G^t)|/|S(A^{(6)}, \mathbb{Z}_n^t)|$.

Let us show the second part of the result for $\mathbf{x} \in S(A^{(6)}, \mathbb{Z}_n^t)$. Given $\mathbf{y}_\kappa \in S(J_\kappa, G_\kappa^t)$, $(\mathbf{y}_\kappa)_i \in G_\kappa^t$, $i \in [1, m^{(j)}]$, denotes the i -th coordinate of \mathbf{y}_κ and $(\mathbf{y}_\kappa)_{i,j} \in G_\kappa$ denotes the (i, j) -th coordinate of the solution with $(i, j) \in [1, m^{(j)}] \times [1, t]$.

Any collection of solutions $\mathbf{y}_\kappa \in S(J_\kappa, G_\kappa^t)$ induce a unique $\mathbf{y} \in (A^{(7)}, G^t)$. The variables indexed in $[k^{(6)} + 1, m^{(6)}]$ parameterize the solutions in $S(A^{(6)}, \mathbb{Z}_n^t)$. Therefore, if we have

$$(\mathbf{x})_i = \sum_{\kappa \in \Upsilon} (\mathbf{y}_\kappa)_{i+1}, \text{ for all } i \in [k^{(6)} + 1, m^{(6)}] \quad (8)$$

for our selected \mathbf{x} , then $\phi(\mathbf{y}) = \mathbf{x}$. The condition (8) is also necessary; if the collection of solutions $\{\mathbf{y}_\kappa\}_{\kappa \in \Upsilon}$ inducing \mathbf{y} does not satisfy (8) for some index $i \in [k^{(6)} + 1, m^{(6)}]$, then $\phi(\mathbf{y}) \neq \mathbf{x}$. The variables that parameterize the solutions for any system (J_κ, G_κ^t) are those indexed in $[k^{(j)} + 1, m^{(j)}]$; once the value of $(\mathbf{y}_\kappa)_i$ is selected for $i \in [k^{(j)} + 1, m^{(j)}]$, the solution $\mathbf{y}_\kappa \in S(J_\kappa, G_\kappa^t)$ exists and is unique. As the proof of Remark 27 highlights, the variable $m^{(j)}$ does not appear in (8); the value of $\phi(\mathbf{y})$ is independent of the values $(\mathbf{y}_\kappa)_{m^{(j)}}$ for $\kappa \in \Upsilon$.

Pick an $i \in [k^{(6)} + 1, m^{(6)}]$ and a value for $(\mathbf{y})_{i+1} \in G^t$ such that $\phi_i((\mathbf{y})_{i+1}) = (\mathbf{x})_i$. All the solutions $\mathbf{y} \in S(A^{(7)}, G^t)$ with $\phi(\mathbf{y}) = \mathbf{x}$ can be found by selecting a value for the remaining parameterizing variables $(\mathbf{y})_{i_1}$ with $i_1 \in [k^{(j)} + 1, m^{(j)}] \setminus \{i+1\}$ appropriately to configure a solution in $S(A^{(7)}, G^t)$ with $\phi(\mathbf{y}) = \mathbf{x}$. For $i_1 \in [k^{(j)} + 1, m^{(j)} - 1] \setminus \{i+1\}$, we can select any $(\mathbf{y})_{i_1} \in G^t$ as long as

$$(\mathbf{x})_{i_1-1} = \sum_{\kappa \in \Upsilon} (\mathbf{y}_\kappa)_{i_1} = \phi_{i_1-1}((\mathbf{y})_{i_1}).$$

²⁰See Section 5.4.

²¹Indeed, $(\bar{J}_\kappa, G_\kappa^t)$ is built from (J_κ, G_κ^t) by adding some variables and the same number of equations in a way that the new equations are: new variable equal linear equation involving old variables. Therefore, a projection onto the right variables using the identity as maps ϕ_i configure the application ϕ of the 1-auto-equivalence.

²²Observe that $S(J_\kappa, G_\kappa^t)$ contains the trivial solution $\mathbf{0}$ and $\{f_\kappa\}_{\kappa \in \Upsilon}$ are homomorphisms.

Additionally, we can select any value for $(\mathbf{y})_{m^{(j)}}$. Observe that the number of choices is independent on the particular value $(\mathbf{y})_{l+1}$ with $\phi_l((\mathbf{y})_{l+1}) = (\mathbf{x})_l$ we have picked.

Given $\mathbf{x} \in S(A^{(6)}, \mathbb{Z}_n^t)$, $i \in [1, k^{(6)}]$ and any $y^{(i)} \in G^t$ with $\phi_i(y^{(i)}) = (\mathbf{x})_i$ we shall find all the solutions $\mathbf{y} \in S(A^{(7)}, G^t)$ with $\phi(\mathbf{y}) = \mathbf{x}$ and $(\mathbf{y})_i = y^{(i)}$. For $\kappa \in [1, m] \times [1, t]$, select any solution $\mathbf{y}'''_{\kappa} \in S(J_{\kappa}, G^t_{\kappa})$ such that

$$(\mathbf{y}'''_{\kappa})_i = y^{(i)}_{\kappa}. \quad (9)$$

Pick an auxiliary solution $\mathbf{y}'_{(1,0)} \in S(J_{(1,0)}, G^t_{(1,0)})$, defined by the last $m^{(j)} - k^{(j)}$ variables,

- choosing a value for $(\mathbf{y}'_{(1,0)})_{m^{(j)}}$ in \mathbb{Z}_n^t (any value.)
- $(\mathbf{y}'_{(1,0)})_{j+1} = (\mathbf{x})_j - \sum_{\kappa \in \Upsilon \setminus \{(1,0)\}} (\mathbf{y}'''_{\kappa})_{j+1}$ for $j \in [k^{(6)} + 1, m^{(6)}]$.

Let \mathbf{y}' be the solution in $S(A^{(7)}, G^t)$ defined by the $mt + 1$ solutions $\{\mathbf{y}'_{(1,0)}, \{\mathbf{y}'''_{\kappa}\}_{\kappa \in \Upsilon \setminus \{(1,0)\}}\}$. Observe that $\phi(\mathbf{y}') = \mathbf{x}$. Indeed, for $j \in [k^{(6)} + 1, m^{(6)}]$,

$$\phi_j(\mathbf{y}') = (\mathbf{y}'_{(1,0)})_{j+1} + \sum_{\kappa \in \Upsilon \setminus \{(1,0)\}} (\mathbf{y}'''_{\kappa})_{j+1} = (\mathbf{x})_j - \sum_{\kappa \in \Upsilon \setminus \{(1,0)\}} (\mathbf{y}'''_{\kappa})_{j+1} + \sum_{\kappa \in \Upsilon \setminus \{(1,0)\}} (\mathbf{y}'''_{\kappa})_{j+1} = (\mathbf{x})_j.$$

Since $\phi(\mathbf{y}') \in S(A^{(6)}, \mathbb{Z}_n^t)$ and \mathbf{x} is unique given the value of its last $m^{(6)} - k^{(6)}$ variables, the claim follows and we have a solution $\mathbf{y}' \in S(A^{(7)}, G^t)$ such that $\phi(\mathbf{y}') = \mathbf{x}$.

However, it is not yet clear that $(\mathbf{y})_i = y^{(i)}$; by (9), we know this equality holds for all coordinates in $\Upsilon \setminus (1,0)$. If $\mathbf{y}'_{(1,0)}$ would be such that $(\mathbf{y}'_{(1,0)})_j = y^{(i)}_{(1,0)}$, then the solution $\mathbf{y}' \in S(A^{(7)}, G^t)$ defined before satisfies the claims $\phi(\mathbf{y}') = \mathbf{x}$ and $(\mathbf{y}')_i = y^{(i)}$. Let us define $\epsilon_{i,j} = \epsilon_{i,j}(\mathbf{x}, \{\mathbf{y}'''_{\kappa}\}_{\kappa \in \Upsilon \setminus \{(1,0)\}}, y^{(i)})$, for $j \in [1, t]$, as the difference between the j -th components of the i -th coordinate of $\mathbf{y}'_{(1,0)}$ and its aimed value $(y^{(i)}_{(1,0)})_j$:

$$(\mathbf{y}'_{(1,0)})_{i,j} - (y^{(i)}_{(1,0)})_j = \epsilon_{i,j}(\mathbf{x}, \{\mathbf{y}'''_{\kappa}\}_{\kappa \in \Upsilon \setminus \{(1,0)\}}, y^{(i)}).$$

Observe that $\epsilon_{i,j} \in G_{i,j}$. Indeed,

$$(\phi_i(y^{(i)}))_j = d_{i,j} \left(\sum_{\kappa \in \Upsilon} (y^{(i)}_{\kappa})_j \right) = (\mathbf{x})_{i,j} \stackrel{\phi(\mathbf{y}') = \mathbf{x}}{=} (\phi_i(\mathbf{y}'))_j = d_{i,j} \left((\mathbf{y}'_{(1,0)})_{i,j} + \sum_{\kappa \in \Upsilon \setminus \{(1,0)\}} (y^{(i)}_{\kappa})_j \right).$$

Therefore $d_{i,j} \epsilon_{i,j} = d_{i,j} \left((\mathbf{y}'_{(1,0)})_{i,j} - (y^{(i)}_{(1,0)})_j \right) = 0$ as claimed.

Using $\epsilon_{i,j}$ we pick, one for each $j \in [1, t]$, a total of t auxiliary solutions $\mathbf{y}^{(j)}_{(1,0)} \in S(J_{(1,0)}, G^t_{(i,j)}) \subset S(J_{(1,0)}, G^t_{(1,0)})$, such that

- $(\mathbf{y}^{(j)}_{(1,0)})_{m^{(j)}} = 0$.
- $(\mathbf{y}^{(j)}_{(1,0)})_{i,j} = \epsilon_{i,j}$.
- $(\mathbf{y}^{(j)}_{(1,0)})_{i,r} = 0$ for $r \in [1, t] \neq j$.²³

These $\{\mathbf{y}^{(j)}_{(1,0)}\}_{j \in [1, t]}$ exist because the greatest common divisor of the coefficients of the $((i-1)t + j)$ -th row of $B^{(3)}_{(1,0)}$ from $J_{(1,0)}$, that define the (i, j) -th variable, is 1.

Consider the collection of t solutions $\{\mathbf{y}''_{(i,j)}\}_{j \in [1, t]}$, $\mathbf{y}''_{(i,j)} \in S(J_{(i,j)}, G^t_{(i,j)})$, that are determined by sharing the last $m^{(j)} - k^{(j)}$ variables with $\mathbf{y}^{(j)}_{(1,0)}$: $(\mathbf{y}''_{(i,j)})_r = (\mathbf{y}^{(j)}_{(1,0)})_r$ for $r \in [k^{(j)} + 1, m^{(j)}]$. Observe that $(\mathbf{y}''_{(i,j)})_i = 0$ since

- $(\mathbf{y}''_{(i,j)})_{i,j} = 0$ as $(\mathbf{y}''_{(i,j)})_{i,j} = (\mathbf{y}''_{(i,j)})_{m^{(j)},1} = (\mathbf{y}^{(j)}_{(1,0)})_{m^{(j)},1} = 0$.

²³There are $|G^t_{(i,j)}|^{m^{(j)} - k^{(j)} - 2}$ such solutions for each $j \in [1, t]$ by Observation 25. We only select one per each j .

- For $r \neq j$, the equations defining $(\mathbf{y}''_{(i,j)})_{i,r}$ and $(\mathbf{y}^{(j)}_{(1,0)})_{i,r}$ from the last $m^{(j)} - k^{(j)}$ variables are the same. Since $(\mathbf{y}^{(j)}_{(1,0)})_{i,r} = 0$, then so is $(\mathbf{y}''_{(i,j)})_{i,r}$.

Let \mathbf{y} be the solution in $S(A^{(\tau)}, G^t)$ formed by

- $\mathbf{y}_\kappa = \mathbf{y}'''_\kappa$ for $\kappa \in ([1, m] \setminus \{i\}) \times [1, t]$.
- $\mathbf{y}_\kappa = \mathbf{y}''_\kappa + \mathbf{y}'''_\kappa$ for $\kappa \in \{i\} \times [1, t]$.
- $\mathbf{y}_{(1,0)} = \mathbf{y}'_{(1,0)} - \sum_{j=1}^t \mathbf{y}^{(j)}_{(1,0)}$.

Observe that

- for $\kappa \in \Upsilon$, $\mathbf{y}_\kappa \in S(J_\kappa, G^t_\kappa)$ as $\mathbf{y}^{(j)}_{(1,0)} \in S(J_{(1,0)}, G^t_{(i,j)}) \subset S(J_{(1,0)}, G^t_{(1,0)})$ and $\mathbf{y}''_{(i,j)} \in S(J_{(i,j)}, G^t_{(i,j)})$ for all $j \in [1, t]$.
- $\phi(\mathbf{y}) = \mathbf{x}$ as $\phi(\mathbf{y}) \in S(A^{(6)}, \mathbb{Z}_m^t)$ and, for $r \in [k^{(6)} + 1, m^{(6)}]$,

$$\begin{aligned} \phi_r(\mathbf{y}) &= (\mathbf{y}_{(1,0)})_{r+1} + \sum_{\kappa \in ([1, m] \setminus \{i\}) \times [1, t]} (\mathbf{y}_\kappa)_{r+1} + \sum_{j \in [1, t]} (\mathbf{y}_{(i,j)})_{r+1} \\ &= (\mathbf{y}'_{(1,0)})_{r+1} - \sum_{j=1}^t (\mathbf{y}^{(j)}_{(1,0)})_{r+1} + \\ &\quad + \sum_{\kappa \in ([1, m] \setminus \{i\}) \times [1, t]} (\mathbf{y}'''_\kappa)_{r+1} + \sum_{j \in [1, t]} \left((\mathbf{y}''_{(i,j)})_{r+1} + (\mathbf{y}'''_{(i,j)})_{r+1} \right) \\ &= (\mathbf{y}'_{(1,0)})_{r+1} + \sum_{\kappa \in \Upsilon \setminus \{(1,0)\}} (\mathbf{y}'''_\kappa)_{r+1} = (\mathbf{x})_r \end{aligned}$$

- $(\mathbf{y})_i = y^{(i)}$. Indeed, for $\kappa \in ([1, m] \setminus \{i\}) \times [1, t]$, $(\mathbf{y}_\kappa)_i = (\mathbf{y}'''_\kappa)_i = y^{(i)}_\kappa$ by hypothesis. Since $(\mathbf{y}''_{(i,j)})_i = 0$, $(\mathbf{y}_{(i,j)})_i = (\mathbf{y}''_{(i,j)})_i + (\mathbf{y}'''_{(i,j)})_i = y^{(i)}_{(i,j)}$ for $j \in [1, t]$. Additionally, for each $j \in [1, t]$,

$$(\mathbf{y}_{(1,0)})_{i,j} = (\mathbf{y}'_{(1,0)})_{i,j} - \sum_{r=1}^t (\mathbf{y}^{(r)}_{(1,0)})_{i,j} = (\mathbf{y}'_{(1,0)})_{i,j} - (\mathbf{y}^{(j)}_{(1,0)})_{i,j} = (\mathbf{y}'_{(1,0)})_{i,j} - \epsilon_{i,j} = (y^{(i)}_{(1,0)})_j.$$

Therefore, the set of solutions $\{\mathbf{y}_\kappa\}_{\kappa \in \Upsilon}$ provides a unique solution \mathbf{y} with $\phi(\mathbf{y}) = \mathbf{x}$ and $(\mathbf{y})_i = y^{(i)}$ as desired. Using Observation 25 on the matrices J_κ , the number of choices made to find \mathbf{y} is independent of the particular $y^{(i)}$. Even more, the excesses $\epsilon_{i,j}$ define a quotient structure among the possible choices of $\{\mathbf{y}'''_\kappa\}_{\kappa \in \Upsilon \setminus \{(1,0)\}}$ and value for $(\mathbf{y}'_{(1,0)})_{m^{(j)}}$ (the other choices for the solutions $\mathbf{y}^{(j)}_{(1,0)}$ can be thought to be fixed depending on $\epsilon_{i,j}$.) This finishes the proof of the remark. \square

Remark 32. Since the system is n -circular and $|G|$ is a divisor of n^α , for some positive integer α , $S_i(A^{(\tau)}, G^t) = G^t$ for all $i \in [1, m^{(\tau)}]$.

5.9 Observation on adding variables

The number of variables in $A^{(\tau)}$, as well as its relative order, is the same as for matrix \overline{J}_κ . This is, the i -th variable in \overline{J}_κ , seen as a system $(\overline{J}_\kappa, G^t_\kappa)$, is one of the coordinates that configure the i -th variable in the system $(A^{(\tau)}, G^t)$.

The hypergraph H used in the representation of $(A^{(\tau)}, G^t)$ can be obtained in the following way. Consider the original $k \times m$ system (A, G_0) . Let H_0 be a $(k+1)$ -uniform hypergraph over the vertex set $V = \{v_1, \dots, v_m\}$ with edges $e_i = \{v_i, \dots, v_{i+k}\}$, indices modulo m . Let us pair the edge e_i with the i -th variable of A , x_i . Let $e_i = \{v_i, \dots, v_{i+k}\}$ and $e_j = \{v_j, \dots, v_{j+k}\}$ be the edges associated with x_i and x_j respectively. The variable x_i is said to be *before* x_j if $v_j \in e_i$. If x_i is before x_j then the variables x_{i+1}, \dots, x_{j-1} are said to be *in-between* x_i and x_j .

Assume that (A', G') is built from (A, G_0) by adding a new variable and a new equation. Let H_1 denote the hypergraph associated with (A', G') as described in the procedure above. Alternatively, H_1 can be constructed from H_0 as follows. When a new variable x_l is added to the system (A, G) , a new vertex v_{m+1} is added to $V(H)$. If a new equation is added, the uniformity of the edges increases by 1. The edges e_1, \dots, e_{l-1} in H_1 start at the same point as the corresponding edges in H_0 and have one more vertex in them. The edges e_{l+1}, \dots, e_{m+1} in H_1 start one vertex later and have one additional vertex than the corresponding ones in H_0 (they finish two vertices later than e_l, \dots, e_m in H_0 .) In particular, if two edges e_i and e_j , with associated variables x_i and x_j , share a vertex in H_0 and x_i is before x_j , then the corresponding edges in H_1 share the same number of vertices if the new variable x_l is between x_i and x_j .²⁴ If the added variable x_l is not between them, then the corresponding edges to e_i and e_j in H_1 share an additional vertex. Moreover, some edges that did not share any vertex in H_0 , may share a vertex in H_1 .

If (A', G') is built from (A, G_0) by adding only a new variable, then H_1 is built from H_0 by adding a new vertex v_{m+1} but not increasing the uniformity of the edges. If x_i is before x_j in H_0 and the new variable x_l is between x_i and x_j , then e_i and e_{j+1} , the edges corresponding to x_i and x_j in (A', G') intersect in one vertex less. Otherwise, the intersection between the new edges does not change with respect to the intersection of the corresponding edges in H_0 .

Observe that the hypergraph H provided by the procedures from Section 5.8 to represent $(A^{(\tau)}, G^t)$ can be thought of as coming from an embryonal cycle H_0 as described above. Let H_2 be the hypergraph obtained from H by removing the edges related to the added variables from (A, G_0) to (J_κ, G_κ^t) . Observe that most of the added variables involve the addition of equations. In these cases, removing the edges from H represents no problems in terms of connectedness of H_2 . However, some variables were added without the addition of any equation. Those free variables correspond to the parts of the construction dealing with $\gamma \neq 1$, Section 5.5, and to simulate the independent vectors in Section 5.2. Since the number of variables used in the simulation of the independent vectors is less than k , the $k+1$ -uniformity of the hypergraph embryo H_0 allows H_2 to be connected.

The remaining cases involve the additional variable added in Section 5.5 to each of the systems (J_κ, G_κ^t) with respect to $(A^{(6)}, \mathbb{Z}_n^t)$. Since $m^{(6)} - k^{(6)} \geq 2$, the procedure from Section 5.7 adds at least four variables between any pair of the first $k^{(j)}$ variables from (J_κ, G_κ^t) . Notice that the first $k^{(6)}$ variables of J_κ contains the original set of m variables of (A, G_0) . Hence, the set of added variables is non-empty and well distributed throughout the original set of variables, with several variables placed in-between original ones and others before, increasing the connectedness. This justifies that H_2 is connected and supported over the same set of vertices as H .

5.10 From the representation of $(A^{(\tau)}, G^t)$ to (A, \mathbb{Z}_n^t)

Let us summarize the steps followed to construct $(A^{(\tau)}, G^t)$ and its relation with the previous systems. Recall that the original homomorphism system $(A', \prod_{i=1}^t \mathbb{Z}_{n_i}) = (A_0, G_0)$ has dimensions $k \times m$. Moreover, we assume that $m \geq k + 2$.

From	To	Relation	Dimensions	Description	In
(A_0, G_0)	(A, \mathbb{Z}_n^t)	μ -equivalent 1	k eq., m var.	From $G_0 = \prod_{i=1}^t \mathbb{Z}_{n_i}$ to \mathbb{Z}_n^t	4.2
(A, \mathbb{Z}_n^t)	$(A^{(1)}, \mathbb{Z}_n^t)$	equivalent	k eq., m var.	row reduction	5.2
$(A^{(1)}, \mathbb{Z}_n^t)$	$(A^{(3)}, \mathbb{Z}_n^t)$	μ -auto-equiv.	k eq., $m + k = m^{(3)}$ var.	from determinantal n to 1, independent vector simulation	5.2
$(A^{(3)}, \mathbb{Z}_n^t)$	$(A^{(4)}, \mathbb{Z}_n^t)$	1-auto-equiv.	$m^{(3)}$ eq., $2m^{(3)} - k^{(3)}$ var.	determinantal 1 to determinant 1	5.3
$(A^{(4)}, \mathbb{Z}_n^t)$	$(A^{(5)}, \mathbb{Z}_n^t)$	equivalent	$m^{(3)} \times 2m^{(3)} - k^{(3)}$	row reduction to $(I_{m^{(3)}} B)$	5.3
$(A^{(5)}, \mathbb{Z}_n^t)$	$(A^{(6)}, \mathbb{Z}_n^t)$	1-auto-equiv.	$k^{(5)}$ eq., $m^{(5)}$ var.	row-reduce t -row blocks in B ; product of gcd of the rows is the determinantal of the block	5.4
$(A^{(6)}, \mathbb{Z}_n^t)$	(J_κ, G_κ^t) , with $\kappa \in \Upsilon$	splitting	$k^{(6)} + 1$ eq., $m^{(6)} + 2$ var. each	find systems $J_\kappa = (I_{k^{(6)}} B)$ with $D_t(B_{[ti+1, ti+t]}) = 1$	5.5
(J_κ, G_κ^t) , with $\kappa \in \Upsilon$	(J_κ, G_κ^t) , with $\kappa \in \Upsilon$	1-auto-equiv.	$k^{(j')}$ eq., $m^{(j')}$ var. each	find n -circular systems for (J_κ, G_κ^t)	5.7
(J_κ, G_κ^t) , with $\kappa \in \Upsilon$	$(A^{(\tau)}, G^t)$	joining	$k^{(j')} = k^{(\tau)}$ eq., $m^{(j')} = m^{(\tau)}$ var.	group the systems (J_κ, G_κ^t) in a single one	5.8
$(A^{(6)}, \mathbb{Z}_n^t)$	$(A^{(\tau)}, G^t)$	μ -equivalent 2	$k^{(\tau)}$ eq., $m^{(\tau)}$ var.	conclusion from joining the systems	5.8

²⁴Here x_l is assumed to be between the variables corresponding to x_i and x_j in H_1 .

Where $k^{(j')} = (4k^{(j)} + 1)(m^{(j)} - k^{(j)})$, $m^{(j')} = (4k^{(j)} + 2)(m^{(j)} - k^{(j)})$, and $G = \prod_{\kappa \in \Upsilon} G_\kappa$.

We prove the first part of Theorem 2 under the conditions $m \geq k + 2$ by concatenating Proposition 15, 14, 13 and 12 between the different pairs of μ -equivalent systems appropriately. For instance we shall use

- Proposition 15 from $(A^{(7)}, G^t)$ to $(A^{(6)}, \mathbb{Z}_n^t)$ by Remark 31.
- Proposition 12 from $(A^{(6)}, \mathbb{Z}_n^t)$ to $(A^{(5)}, \mathbb{Z}_n^t)$ by Remark 26.
- No proposition is needed from $(A^{(5)}, \mathbb{Z}_n^t)$ to $(A^{(4)}, \mathbb{Z}_n^t)$ as they are equivalent.
- Proposition 12 from $(A^{(4)}, \mathbb{Z}_n^t)$ to $(A^{(3)}, \mathbb{Z}_n^t)$ by Remark 24.
- Proposition 13 from $(A^{(3)}, \mathbb{Z}_n^t)$ to $(A^{(1)}, \mathbb{Z}_n^t)$ by Remark 22.
- Proposition 14 from $(A^{(1)}, \mathbb{Z}_n^t)$ to (A_0, G_0) by Remark 18 (with $A^{(1)} = A'$ and $(A_0, G_0) = (A, G)$).

These propositions can be concatenated by Section 5.9. Indeed, the edges related to the remaining variables, after the composition of the maps defining the μ -equivalences, still cover all the vertices of the initial hypergraph H . Thus this shows the following proposition.

Proposition 33 (Representation for homomorphisms). *Let G be a finite abelian group and let m, k be two positive integers with $m \geq k + 2$. Let A be a homomorphism $A : G^m \rightarrow G^k$ and let $\mathbf{b} \in G^k$ be given. Then the system of linear configurations $((A, \mathbf{b}), G)$ is γ -strongly-representable with $\gamma_i = |G|/|S_i(A, G)|$ and where χ_1, χ_2 depend only on m .*

By means of Theorem 1, Proposition 33 proves the first part of Theorem 2 when $m \geq k + 2$. The second part of Theorem 2 and the treatment of the cases when $m < k + 2$ are proved in Section 6.

6 Proof of Theorem 2: second part and the cases $m < k + 2$

The cases where $k > m$ can be reduced to $m = k$ by eliminating the redundant equations (for instance, thinking of them as equations in \mathbb{Z}_n using Section 4.2 and Section 5.1 and performing Gaussian elimination on the matrices only allowing integer operations.)

Let G_0 be a finite abelian group. If the system $A\mathbf{x} = 0$, $\mathbf{x} \in \prod_{i=1}^m X_i$, with $|S(A, G_0, \prod_{i=1}^m X_i)| < \delta|S(A, G_0)|$ has k equations and m variables, then $A'\mathbf{x} = 0$, $\mathbf{x} \in \prod_{i=1}^m X_i \times G_0^2$, with

$$A' = \left(A \left| \begin{array}{cc} |G_0| & |G_0| \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{array} \right. \right)$$

has k equations in $m + 2$ variables and $|S(A', G_0, \prod_{i=1}^m X_i \times G_0^2)| < \delta|S(A', G_0)|$. Therefore, the cases $m < k + 2$ can be proved using the second part of Theorem 2 restricted to the case $m \geq k + 2$. This is encapsulated in Observation 34.

Observation 34 (Do not remove from full sets). *Assume that the γ -representation of the $k \times m$ homomorphism system (A, G_0) , with $m \geq k + 2$, is constructed using a system with an n -circular matrix $(A^{(7)}, G)$ by the methods exposed in sections 4.1 through 5.10. Let $I \subset [1, m]$ be a set of indices such that $X_i = G_0$, $i \in I$. Then Theorem 2 holds with $X'_i = \emptyset$.*

Proof of Observation 34. Suppose $m \geq k + 2$. By reordering the variables, assume that $I = [j, m]$ for some j . Represent the system (A, G_0) by the s -uniform hypergraph pair (K_0, H) . By the construction in Section 5.8 and Section 5.10 and the comments in Section 5.9, H is connected and K_0 is $|H|$ -partite. Let $V_1, \dots, V_{|H|}$ denote the stable sets in K_0 and recall that $|V_i| = |V_j|$ for $i, j \in [1, |H|]$. Let $K = K_0 (\prod_{i=1}^m X_i)$ represent the hypergraph where only the edges labelled $x_i \in X_i$ and colored i appear. Let K' and H' be the hypergraphs obtained by removing the edges colored $i \in [j, m]$ from K and H respectively. Assume $e_1 = \{1, \dots, s\} \subset H$ is the edge related to x_1 .

Let d be the number of vertices in H' with no edges. By the construction of the representation from the n -circular matrix provided in the third part of Section 5.8 and knowing that the deleted edges have the largest indices, the stable sets V_i (clusters of vertices) of K' with no edge incident with them are the consecutive ones $V_{|H|-d+1}, \dots, V_{|H|}$. Since every subset of s clusters from K' span, at most, one color class of edges, the connected component of H' has one vertex in each of the first $|H|-d$ clusters. The isolated vertices of H' can be placed in any cluster $V_i, i \in [1, |H|]$. Thus, each copy of H in K ($\prod_{i=1}^m X_i$) generates $\binom{|K|-(|H|-d)}{d} \approx c'|K|^d$ copies of H' in K' , for some constant c' .

Let H'' be the hypergraph built from H' by removing its isolated vertices. Let K'' be built from K' by removing the isolated stables $V_{|H|-d+1}, \dots, V_{|H|}$. Then (K'', H'') is a representation of the system (A, G_0) where (x_1, \dots, x_{j-1}) is a solution if and only if there is some $(x_j, \dots, x_m) \in G_0^{m-j+1}$ for which $(x_1, \dots, x_{j-1}, x_j, \dots, x_m)$ is a solution. Indeed, we can extend any copy H'' in K'' to $\left| \prod_{i=|H|-d+1}^{|H|} V_i \right| = |V_i|^d = c|K|^d$ copies of H in K by selecting one additional vertex in each of the last d clusters. Since $X_i = G_0$, K has all the possible edges coloured $i \in I$ in the vertices $\{(\cdot, j)\}_{j \in \{i_1, \dots, i_s\}}$, where $\{i_1, \dots, i_s\}$ is the support of the edge coloured i in H . Therefore, any choice in $\prod_{i=|H|-d+1}^{|H|} V_i$ completes a copy of H'' in K'' to a copy of H in K . On the other hand, any copy of H in K generates one copy of H'' in K'' . Thus,

$$|C(H, K)| = |C(H'', K'')| |V_i|^d = |C(H'', K'')| c|K|^d$$

and the proportions $|K|^{|H|}/|C(H, K)|$ and $|K''|^{|H''|}/|C(H'', K'')|$ are such that

$$\frac{|K|^{|H|}}{|C(H, K)|} = \frac{|K|^{|H|}}{c|K|^d|C(H'', K'')|} = c'' \frac{|K''|^{|H''|}}{|C(H'', K'')|}$$

where c'' only depends on m .

Therefore, if $|S(A, G_0, \prod_{i=1}^{j-1} X_i \times G_0^{m-j+1})| < \delta |S(A, G_0)|$, then, by the representability for (A, G_0) by (K, H) , $|C(H, K)| < \delta' |K|^{|H|}$. Hence $|C(H'', K'')| < \delta'' |K''|^{|H''|}$. By applying the same procedure of Theorem 1 to (K'', H'') , we show a removal lemma by obtaining sets $X'_i \subset X_i, i \in [1, j-1]$, as those are the represented variables, such that $\prod_{i=1}^{j-1} X_i \setminus X'_i$ has no solution (for any value of the last variables.) Therefore, we obtain the additional property that $X'_i = \emptyset$ for $i \in [j, m]$. \square

Let us observe that Observation 34 can be used to obtain a similar additional conclusion for [21, Theorem 1].

7 Conclusions and final comments

In this paper we have presented Definition 10, a notion of representation of a system of configurations using a pair of hypergraphs that generalizes previous definitions. Additionally, the notion is strong enough to translate the combinatorial removal lemma, Theorem 11, to the context of system of configurations, Theorem 1. We observe that the systems of configurations induced by “copies of a hypergraph H in K ” is representable. Additionally, we present a representation for the systems induced by “patterns of the permutation τ in the permutation σ ”.

The extra flexibility given by Definition 10 with respect to previous notions allows us to show that the configuration systems defined by homomorphisms between finite abelian groups are representable (see Proposition 33). The combination of Proposition 33 and Theorem 1 is reflected in the removal lemma for homomorphism systems of finite abelian groups, Theorem 2, which is the main result of the paper. Several applications of Theorem 2 are given in the introduction.

In [5], a more algebraic definition of representation tied with the infinite aspect of the compact abelian groups is presented as [5, Definition 3.7], which follows the lines of [34]. Indeed, the strong version of Definition 10 can be seen as the discrete and combinatorial analogue of [5, Definition 3.7]. Let us mention that the construction of the representation for the homomorphisms of finite abelian groups presented in sections 4-6 can be adapted to fit [5, Definition 3.7]. It is natural to ask if a removal lemma result holds for homomorphisms in compact abelian groups. However, such result presents technical difficulties involving some aspects of the construction presented here that arise when generic compact abelian groups are considered.²⁵

²⁵These difficulties are related with the extension of matrices performed in Section 5.7.

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